Chapter 4

Primitive equations in pressure coordinates

The primitive equations serve as the fundamental set of equations in this part of the lecture. The present chapter introduces pressure as new vertical coordinate, because this turns out convenient for the subsequent development.

4.1 Primitive equations

We want to describe synoptic or larger scale flow of a dry atmosphere on the β plane. Using rectangular cartesian coordinates the equations read

$$\boxed{\begin{aligned} &\frac{D\mathbf{v}}{Dt} + f\mathbf{k}\times\mathbf{v} = -\frac{1}{\rho}\nabla_h p + \mathbf{X}\ , && (a)\\ &0 = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}\ , && (b)\\ &c_p\frac{D\theta}{Dt} = \left(\frac{p_0}{p}\right)^{\kappa} J\ , && (c)\\ &\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0\ , && (d)\end{aligned}} \tag{4.1}$$

Here, $\mathbf{u} = (u, v, w)$ denotes the three-dimensional wind, $\mathbf{v} = (u, v)$ is the horizontal wind, $D/Dt = \partial_t + \mathbf{u}\cdot\nabla$ is the material derivative following the three-dimensional flow, $\nabla = (\partial_x, \partial_y, \partial_z)$ is the three-dimensional gradient operator, $\nabla_h = (\partial_x, \partial_y)$ is the horizontal gradient operator, p is pressure, ρ is density, $f = f_0 + \beta y$ is the Coriolis parameter with constant f_0 and β, g is the constant acceleration due to gravity, c_p is the specific heat at constant pressure, $\kappa = R/c_p$, R is the gas constant for dry air, J is the diabatic heating rate per unit mass, $\mathbf{X}$ is a nonconservative force in the equations for horizontal momentum (typically friction), and $\mathbf{k}$ is the unit vector in the vertical. Potential temperature θ and temperature T are related through

$$\boxed{\theta = T\left(\frac{p_0}{p}\right)^{\kappa}} \tag{4.2}$$

where p_0 is a constant reference pressure (often taken to be $p_0 = 1000$ hPa). In addition the equation of state for a dry atmosphere can be approximated by the ideal gas law

$$\boxed{p = \rho R T}\ . \tag{4.3}$$

The above set of equations implies two major approximations: first the hydrostatic approximation (viz. (4.1b)), and second the Coriolis force in (4.1a) being calculated from the horizontal wind components $\mathbf{v}$ only. On this level of approximation the equations are called *primitive equations*.

If we consider the nonconservative terms $\mathbf{X}$ and J as given, we have seven equations for the seven unknowns $\mathbf{u} = (u, v, w)$, T, θ, p, and ρ. Both the ideal gas law and the definition of potential temperature are simple algebraic relation between p, ρ, T and θ and can be used to eliminate two of these four variables, leaving five equations for five unknowns. One possible way of reduction is as follows

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k}\times\mathbf{v} = -\frac{1}{\rho}\nabla_h p + \mathbf{X}\ , \tag{4.4}$$

$$0 = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}\ , \tag{4.5}$$

$$\frac{D}{Dt}\left(\frac{p^{1-\kappa}}{\rho}\right) = \frac{\kappa}{p^{\kappa}}J\ , \tag{4.6}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0\ , \tag{4.7}$$

which involves the variables $\mathbf{u} = (u, v, w)$, p and ρ.

One of these remaining equations is diagnostic, namely (4.5). Had we not made the hydrostatic approximation this equation would read $\partial w/\partial t + \ldots$. Hence, we have lost the equation for the vertical wind owing to the hydrostatic approximation. Note, that w is implicitly contained in the operator D/Dt and must, therefore, be part of a complete solution. In addition, in practical meteorological applications the vertical wind is an important field, since upwelling often leads to formation of clouds and rain, while downwelling make clouds disappear. It is unfortunate that the very variable which our customers are interested in has almost disappeared from the basic set of equations. In the following we shall come back to this issue several times and present more or less indirect methods that allow us to determine the vertical wind.

4.2 Pressure as vertical coordinate

The hydrostatic equation (4.1b) can be used to express the amount of mass m per unit surface contained between altitude z_1 and z_2 as a function of the pressure difference between these two levels,

$$m = \int_{z_1}^{z_2} \rho \, dz = -\int_{p_1}^{p_2} \frac{dp}{g} = \frac{p_1 - p_2}{g} \,, \tag{4.8}$$

where pressure p_i corresponds to altitude z_i. For $z_2 \to \infty$ we get

$$\boxed{m = \int_{z_0}^{\infty} \rho \, dz = \frac{p_0}{g}} \,. \tag{4.9}$$

In a hydrostatic atmosphere pressure is simply a measure for the amount of mass per unit area contained in a column of air overhead.

Writing the hydrostatic relation as

$$\frac{\partial p}{\partial z} = -g\rho \,, \tag{4.10}$$

and using the equation of state for an ideal gas, this can be integrated to give

$$\boxed{p(z) = p_0 \, e^{-\int_{z_0}^{z} \frac{g}{RT} \, dz}} \,, \tag{4.11}$$

where $p_0 = p(z_0)$. Since we can always assume $T > 0$, we found that for (x, y) kept fixed there is a unique relation between z and p.

Coordinate transformation

It follows that p can be used as vertical coordinate instead of z. Formally we have a coordinate transformation

$$x(x^*, y^*, p^*) = x^* \,, \tag{4.12}$$
$$y(x^*, y^*, p^*) = y^* \,, \tag{4.13}$$
$$z(x^*, y^*, p^*) = -H \ln(p^*/p_0) \,, \tag{4.14}$$

and the respective back transformation

$$x^*(x, y, z) = x \,, \tag{4.15}$$
$$y^*(x, y, z) = y \,, \tag{4.16}$$
$$p^*(x, y, z) = p_0 \exp(z/H) \,. \tag{4.17}$$

For the transformation to be nonsingular we need to require that the Jacobian be nonzero, i.e.

$$J = \frac{\partial(x, y, z)}{\partial(x^*, y^*, p^*)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \partial z/\partial p^* \end{vmatrix} \neq 0 \,. \tag{4.18}$$

This poses no difficulty in the present case, because $J = \partial z/\partial p^* = -(g\rho)^{-1} \neq 0$ is satisfied always. In the following we drop the asterisk distinguishing the new from the original coordinates (but care must be exercised when interpreting partial derivatives, see below).

Note the reversed roles of z and p in the two coordinate systems: using z as a vertical coordinate, pressure p is an dependent variable with $p(z)$ indicating the pressue at altitude z; on the other hand, using p as vertical coordinate, altitude z is a dependent variable with $z(p)$ indicating the altitude of a certain pressure surface.

Consider a scalar field Ψ being a function of space and time. No matter what coordinates we use to describe space, the value of Ψ at a certain point $\mathbf{x}$ must not depend on the choice of coordinates, i.e.

$$\Psi(x, y, z, t) = \tilde{\Psi}(x, y, p, t) \ , \tag{4.19}$$

where z corresponds to p according to the coordinate transformation. The left hand side uses geometric height as vertical coordinate while the right hand side uses pressure as vertical coordinate. We introduced different symbols Ψ and $\tilde{\Psi}$ to indicate that the functional relationship between the scalar field and the coordinates is different in the two coordinate systems. As we shall see, *pressure as vertical coordinate* has significant advantages over z as vertical coordinate, but we do not get these advantages for free. For (x, y, p) is *not* a rectangular coordinate system any longer and care has to be exercised when performing the coordinate tranformation. One way to proceed would be by using the rules for transformations to general curvilinear coordinate systems involving the metric tensor. Here we shall proceed in a somewhat heuristic manner, which is less rigorous but more intuitive.

The material derivative of a scalar quantity in geometric coordinates, $\Psi = \Psi(x, y, z, t)$ is

$$\begin{aligned} \frac{D\Psi}{Dt} &= \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial x}\frac{Dx}{Dt} + \frac{\partial \Psi}{\partial y}\frac{Dy}{Dt} + \frac{\partial \Psi}{\partial z}\frac{Dz}{Dt} = \\ &= \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\Psi \ , \end{aligned} \tag{4.20}$$

where $u = Dx/Dt$, $v = Dy/Dt$, and $w = Dz/Dt$. In pressure coordinates we have $\tilde{\Psi} = \tilde{\Psi}(x, y, p, t)$ and we get

$$\begin{aligned} \frac{D\tilde{\Psi}}{Dt} &= \frac{\partial \tilde{\Psi}}{\partial t} + \frac{\partial \tilde{\Psi}}{\partial x}\frac{Dx}{Dt} + \frac{\partial \tilde{\Psi}}{\partial y}\frac{Dy}{Dt} + \frac{\partial \tilde{\Psi}}{\partial p}\frac{Dp}{Dt} = \\ &= \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p}\right)\tilde{\Psi} \ , \end{aligned} \tag{4.21}$$

where now we set

$$u = \frac{Dx}{Dt} \ , \quad v = \frac{Dy}{Dt} \ , \quad \omega = \frac{Dp}{Dt} \ . \tag{4.22}$$

It is important to note that the horizontal gradient has a different meaning in the different coordinate systems: for geometric coordinates it means the rate of change with x and y with z kept fixed, while for pressure coordinates it means the rate of change with x and y with p kept fixed.

Let $\mathbf{x}$ denote the location of a fluid element. Using geometric coordinates, the three-dimensional wind is given through

$$\mathbf{u} = \frac{D\mathbf{x}}{Dt} = \frac{\partial \mathbf{x}}{\partial x}\frac{Dx}{Dt} + \frac{\partial \mathbf{x}}{\partial y}\frac{Dy}{Dt} + \frac{\partial \mathbf{x}}{\partial z}\frac{Dz}{Dt} = u\,\mathbf{i} + v\,\mathbf{j} + w\,\mathbf{k} \ , \tag{4.23}$$

where again $u = Dx/Dt$, $v = Dy/Dt$, and $w = Dz/Dt$ and where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectores in the three mutually perpendicular coordinate directions. Doing the same with pressure as vertical coordinate, we obtain

$$\mathbf{u} = \frac{D\mathbf{x}}{Dt} = \frac{\partial \mathbf{x}}{\partial x}\frac{Dx}{Dt} + \frac{\partial \mathbf{x}}{\partial y}\frac{Dy}{Dt} + \frac{\partial \mathbf{x}}{\partial p}\frac{Dp}{Dt} = u\,\boldsymbol{\tau}_{(x)} + v\,\boldsymbol{\tau}_{(y)} + \omega\,\boldsymbol{\tau}_{(p)}\ , \tag{4.24}$$

where again $u = Dx/Dt$, $v = Dy/Dt$ and $\omega = Dp/Dt$ and where

$$\boldsymbol{\tau}_{(\sigma)} = \frac{\partial \mathbf{x}}{\partial \sigma} \tag{4.25}$$

(with σ denoting any of the three coordinates x, y or p) are the three vectors along the three coordinate directions, which generally are *not* mutually orthogonal any longer. Note that $\boldsymbol{\tau}_{(x)}$ and $\boldsymbol{\tau}_{(y)}$ may generally have a component in the vertical direction and are in this sense not purely horizontal.

The gradient of a scalar function $\tilde{\Psi}(x, y, p)$ is

$$\nabla\tilde{\Psi} = \frac{\partial\tilde{\Psi}}{\partial x}\nabla x + \frac{\partial\tilde{\Psi}}{\partial y}\nabla y + \frac{\partial\tilde{\Psi}}{\partial p}\nabla p = \frac{\partial\tilde{\Psi}}{\partial x}\boldsymbol{\eta}_{(x)} + \frac{\partial\tilde{\Psi}}{\partial y}\boldsymbol{\eta}_{(y)} + \frac{\partial\tilde{\Psi}}{\partial p}\boldsymbol{\eta}_{(p)}\ , \tag{4.26}$$

where the vectors $\boldsymbol{\eta}_{(\sigma)}$ (with $\sigma = x, y, p$) are perpendicular to the coordinate surfaces. Note that generally neither $\boldsymbol{\eta}_{(\sigma)}$ nor $\boldsymbol{\tau}_{(\sigma)}$ form an orthogonal system. Both can be used to span the three-dimensional space, but they are distinct from each other. If we want to describe a vector in terms of its components along the direction of basis vectors, we have a choice, which leads to the distinction between covariant and contravariant components. The two sets of basic vectors $\boldsymbol{\eta}_{(\sigma)}$ and $\boldsymbol{\tau}_{(\sigma)}$ are dual in the sense that

$$\boldsymbol{\eta}_i \cdot \boldsymbol{\tau}_j = \delta_{ij}\ , \tag{4.27}$$

where now the index i and j represent the three coordinates x, y, and p.

Transformation of the primitive equations

First we outline a simple heuristic derivation of the continuity equation in pressure coordinates. Consider a material fluid element of mass δm. Conservation of mass refers to material conservation and can, therefore, be written as

$$\frac{D}{Dt}\,\delta m = 0\ . \tag{4.28}$$

Using the hydrostatic relation we can write δm as

$$\delta m = \rho\,\delta x\,\delta y\,\delta z = g^{-1}\,\delta x\,\delta y\,\delta p\ , \tag{4.29}$$

and mass conservation becomes

$$\begin{aligned}\frac{D}{Dt}\left(g^{-1}\delta x\,\delta y\,\delta p\right) &= \frac{1}{g}\left(\delta u\,\delta y\,\delta p + \delta x\,\delta v\,\delta p + \delta x\,\delta y\,\delta\omega\right) = \\ \frac{\delta x\,\delta y\,\delta p}{g}&\left(\frac{\delta u\,\delta y\,\delta p}{\delta x\,\delta y\,\delta p} + \frac{\delta x\,\delta v\,\delta p}{\delta x\,\delta y\,\delta p} + \frac{\delta x\,\delta y\,\delta\omega}{\delta x\,\delta y\,\delta p}\right) = \\ &= \delta m\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p}\right) = 0\ ,\end{aligned} \tag{4.30}$$

which immediately yields the *continuity equation in pressure coordinates*

$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0} . \tag{4.31}$$

The continuity equation in pressure coordinates is much simpler than its corresponing version in geometric coordinates. Equation (4.31) is a linear diagnostic relation describing a simple constraint regarding the three components of the wind. On the other hand, the original continuity equation is prognostic and nonlinear and, therefore, more difficult to deal with generally.

Next we introduce the *geopotential* Φ as

$$\boxed{\Phi = gz} . \tag{4.32}$$

Note that in pressure coordinates $\Phi = \Phi(x, y, p) = gz(x, y, p)$, and $\Phi(x, y, p)$ can be interpreted as g times the altitude of surface of pressure p. Often this is abbreviated as $\Phi_{(p)}$, like for instance Φ_{850} for the pressure surface $p = 850$ hPa. Rewriting the hydrostatic relation (4.1b) as

$$\frac{\partial z}{\partial p} = -\frac{1}{g\rho} , \tag{4.33}$$

we get

$$\boxed{\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\frac{RT}{p} = -\frac{R}{p}\left(\frac{p}{p_0}\right)^{\kappa} \theta} , \tag{4.34}$$

where in the last two steps we used the equation of state for an ideal gas and the definition of potential temperature. Thus, in pressure coordinates the hydrostatic relation becomes an equation involving the pressure (i.e. "vertical") derivative of geopotential.

Finally we have to transform the two equations for the components of the horizontal wind **v**. For this we have to learn how to transform the components of a gradient. For a scalar field we can write

$$\Psi(x, y, z) = \tilde{\Psi}(x, y, p[x, y, z]) , \tag{4.35}$$

and the partial derivative with respect to a horizontal coordinate s (with $s = x, y$) becomes

$$\left.\frac{\partial \Psi}{\partial s}\right|_z = \left.\frac{\partial \tilde{\Psi}}{\partial s}\right|_p + \frac{\partial \tilde{\Psi}}{\partial p} \left.\frac{\partial p}{\partial s}\right|_z . \tag{4.36}$$

For clarity we indicated which variables are fixed during partial differentiation. Choosing geometric height as the value of the scalar function, the above reduces to

$$0 = \left.\frac{\partial z}{\partial s}\right|_p + \frac{\partial z}{\partial p} \left.\frac{\partial p}{\partial s}\right|_z \tag{4.37}$$

or

$$\left.\frac{\partial p}{\partial s}\right|_z = -\frac{\partial z/\partial s|_p}{\partial z/\partial p} = +g\rho \left.\frac{\partial z}{\partial s}\right|_p , \tag{4.38}$$

where in the last step we used the hydrostatic relation (4.1b). Introducing geopotential $\Phi = gz$, we obtain the following expression for the horizontal pressure force in pressure coordinates

$$\boxed{-\frac{1}{\rho}\nabla_h p\Big|_z = -\nabla_h \Phi|_p} \ . \tag{4.39}$$

So the equations for the two components u and v of the horizontal wind become

$$\frac{Du}{Dt} - fv = -\partial_x \Phi + X \ , \tag{4.40}$$

$$\frac{Dv}{Dt} + fu = -\partial_y \Phi + Y \ , \tag{4.41}$$

where it is understood that the partial derivatives have to be taken with p kept fixed. Defining the *horizontal wind* $\mathbf{v}$ as the projection of the three-dimensional wind $\mathbf{u}$ onto the local horizontal,

$$\mathbf{v} = u\,\mathbf{i} + v\,\mathbf{j} \ , \tag{4.42}$$

the equation for the horizontal components of momentum can formally be written as

$$\frac{D\mathbf{v}}{Dt} + f\mathbf{k} \times \mathbf{v} = -\nabla_h \Phi + \mathbf{X} \ . \tag{4.43}$$

Note, however, that the threedimensional wind $\mathbf{u}$ may have a vertical component even for $\omega = 0$ owing to (4.24).

Summarizing the above, the primitive equations in pressure coordinates read

$$\boxed{\begin{aligned}
&\frac{D\mathbf{v}}{Dt} + f\mathbf{k} \times \mathbf{v} = -\nabla_h \Phi + \mathbf{X} \ , && (a)\\
&\frac{\partial \Phi}{\partial p} = -\frac{1}{\rho} = -\frac{R}{p}\left(\frac{p}{p_0}\right)^{\kappa} \theta \ , && (b)\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \ , && (c)\\
&c_p \frac{D\theta}{Dt} = \left(\frac{p_0}{p}\right)^{\kappa} J \ . && (d)
\end{aligned}} \tag{4.44}$$

with $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + \omega\partial/\partial p$. Given J and $\mathbf{X}$, the above are five equations for the five unknown fields $\mathbf{u} = (u, v, w), \Phi$ and θ, all being functions of (x, y, p, t).

The major advantages of this set of equations in comparison with the original one are

- the continuity equation (4.44c) is linear and corresponds to a simple constraint for u, v and ω,
- the pressure force in the horizontal momentum equations is linear and a simple horizontal gradient of Φ, rather than the product of a horizontal gradient of p with ρ^{-1}.

To some it may appear strange that the momentum equations have obtained a simpler form by transforming them to non-orthogonal coordinates. Indeed, nonorthogonal coordinates usually give rise to extra terms for the components of a vector equation, so why did we get a simplification in our case? The solution to this question is the fact that our transformation only affects the vertical coordinate, and this mostly affects the vertical component of the equation. The latter, however, is not taken in its mathematically correct form, but rather replaced by a meteorological approximation (namely the hydrostatic approximation). More detail and the formal derivation using the transformation to generalized coordinates can be found in e.g. Dutton (1995).

Sigma coordinates

There is a slight disadvantage using pressure instead of geometric coordinates, and this is associated with the location of boundaries and, hence, the formulation of boundary conditions. Assume that the lower boundary is at $z = 0$ or $z = h_T(x, y)$, where h_T is a known function of x and y. This means that the lower boundary is at a known and fixed location within the geometric coordinate system. On the other hand, in pressure coordinates the surface is located at $p = p_s(x, y, t)$, where p_s denotes the surface pressure. Unfortunately p_s is not known and fixed, rather it is part of the solution and generally varies with time.[1] As a way out one can transform the equations using a new vertical coordinate σ, which is defined as a scaled pressure according to

$$\sigma = \frac{p}{p_s} \, . \tag{4.45}$$

Now the upper and lower surface correspond to $\sigma = 0$ and $\sigma = 1$, respectively, and these values are fixed and known a priori. Using these so-called *sigma coordinates* has proven useful in an actual implementations of the equations for numerical weather prediction.

4.3 Log-p coordinates

There is another disadvantage of pressure coordinates. Layers of constant Δp become increaslingly thick geometrically (corresponding to larger Δz) as one moves upward in the atmosphere,

$$\Delta z = \frac{\Delta p}{g\rho} \, , \tag{4.46}$$

because ρ decreases approximately exponentially with altitude. In other words, p as vertical coordinate is *not height-like*. For instance, a wave with constant vertical wavelength $\exp(imz)$ $(m = const)$ has a more complicated structure in pressure coordinates, namely $\exp(i\tilde{m}p)$ with $\tilde{m}$ not being constant. As a way out we now introduce $\ln p$ as vertical coordinate instead of p. The mentioned disadvantage disappears while all advantages of pressure coordinates are kept.

[1] This is known to anybody who has ever used a barometer at a fixed location close to the surface: as time goes by the pressure varies!

More specifically we define

$$\boxed{z^* := -H \ln \frac{p}{p_0}} \,, \tag{4.47}$$

where H is a constant scale height and p_0 a constant reference pressure. Usually $p_0 = 1000$ hPa is chosen such that $z^* = 0$ corresponds to $p = 1000$ hPa. By definition z^* is a unique function of p, so z^* is really a rescaled pressure coordinate, and we can make use of most of the work we did earlier in order to derive the equations in ln-p coordinates. The full coordinate transformation is

$$x^*(x, y, p) = x \,, \tag{4.48}$$
$$y^*(x, y, p) = y \,, \tag{4.49}$$
$$z^*(x, y, p) = -H \ln \frac{p}{p_0} \,. \tag{4.50}$$

The reverse transformation is

$$x(x^*, y^*, z^*) = x^* \,, \tag{4.51}$$
$$y(x^*, y^*, z^*) = y^* \,, \tag{4.52}$$
$$p(x^*, y^*, z^*) = p_0 \, e^{-z^*/H} \,. \tag{4.53}$$

The Jacobian of this transformation is

$$\frac{\partial(x^*, y^*, z^*)}{\partial(x, y, p)} = \frac{\partial z^*}{\partial p} = \frac{dz^*}{dp} = -\frac{H}{p} \,. \tag{4.54}$$

Comparison with (4.11) shows that in an isothermal atmosphere (i.e. for $T = const$) our new coordinate z^* is equal to geometric altitude and the scale height H is given by RT/g. In this sense z^* is *height-like*. We define a reference temperature T_s through

$$gH = RT_s \,. \tag{4.55}$$

For later reference we note that

$$\frac{dp}{dz^*} = -\frac{p}{H} \quad \text{and} \quad \frac{dz^*}{dp} = -\frac{H}{p} \,. \tag{4.56}$$

If we consider $z^* = z^*(x, y, z)$, we get

$$\frac{\partial z^*}{\partial z} = \frac{dz^*}{dp}\frac{\partial p}{\partial z} = -\frac{H}{p}(-g\rho) = +\frac{gH}{RT} = \frac{T_s}{T} \,, \tag{4.57}$$

where we used the hydrostatic relation, the ideal gas law, and the definition of T_s. This equation, i.e.

$$\boxed{\frac{\partial z^*}{\partial z} = \frac{T_s}{T}} \tag{4.58}$$

makes more explicit the statemtent that z^* differs from z in the sense as the actual temperature T differes from the constant reference temperature T_s. For later reference we note that the term $(p_0/p)^\kappa$, which appears in the pressure coordinate version of the primitive equations, becomes

$$\left(\frac{p_0}{p}\right)^\kappa = e^{\frac{\kappa z^*}{H}} \tag{4.59}$$

in ln-p coordinates. As a consequence, the potential temperature is related to temperature as follows

$$\boxed{\theta = T\, e^{\frac{\kappa z^*}{H}}} \ . \tag{4.60}$$

First we introduce

$$\Phi(x,y,p) = \tilde{\Phi}(x^*,y^*,z^*) \tag{4.61}$$

and rewrite the hydrostatic relation (4.44b) as

$$\frac{\partial \Phi}{\partial p} = \frac{\partial \tilde{\Phi}}{\partial z^*}\frac{dz^*}{dp} = \frac{\partial \tilde{\Phi}}{\partial z^*}\left(-\frac{H}{p}\right) = -\frac{1}{\rho} \ , \tag{4.62}$$

which becomes

$$\frac{\partial \tilde{\Phi}}{\partial z^*} = \frac{p}{\rho H} = \frac{RT}{H} = g\frac{T}{T_s} \tag{4.63}$$

or

$$\boxed{\frac{\partial \tilde{\Phi}}{\partial z^*} = g\frac{T}{T_s}} \ . \tag{4.64}$$

The change of geopotential with ln-p altitude is given by the temperature of the atmosphere. If consider an atmospheric layer between two given pressure surfaces which are characterized by Δz^* (e.g. the 1000-500 hPa relative tropography), this equation tells us that the layer's thickness Δz in geometric space is proportional to the mean temperature of that layer

$$\frac{g\Delta z}{\Delta z^*} = \frac{g}{T_s}T \quad \text{or} \quad \Delta z \propto T \ . \tag{4.65}$$

The natural definition of the vertical wind in ln-p coordinates is

$$\boxed{w = \frac{Dz^*}{Dt}} \ , \tag{4.66}$$

and we obtain

$$w = \frac{dz^*}{dp}\frac{Dp}{Dt} = -\frac{H}{p}\omega \ . \tag{4.67}$$

This allows us to rewrite the continuity equation (4.44c). The third term on the left hand side becomes

$$\frac{\partial \omega}{\partial p} = \frac{\partial \tilde{\omega}}{\partial z^*}\frac{dz^*}{dp} = -\frac{H}{p}\frac{\partial \tilde{\omega}}{\partial z^*} = -\frac{H}{p}\frac{\partial}{\partial z^*}\left(-\frac{p}{H}\tilde{w}\right) = \frac{1}{p}\frac{\partial}{\partial z^*}(p\tilde{w}) \ , \tag{4.68}$$

and the full continuity equation reads

$$\frac{\partial \tilde{u}}{\partial x^*} + \frac{\partial \tilde{v}}{\partial y^*} + \frac{1}{p}\frac{\partial}{\partial z^*}(p\tilde{w}) = 0 \ . \tag{4.69}$$

We introduce the *pseudo density*

$$\boxed{\rho_0(z^*) = \frac{p(z^*)}{RT_s} = \frac{p_0}{RT_s}\, e^{-\frac{z^*}{H}}}\,, \tag{4.70}$$

which plays the same role in ln-p coordinates as the actual density ρ in geometric coordinates. For, consider the mass of a thin column of unit surface

$$dm = \rho\, dz = \rho\, \frac{\partial(x, y, z)}{\partial(x^*, y^*, z^*)}\, dz^* = \frac{p}{RT}\frac{\partial z}{\partial z^*}\, dz^* = \frac{p}{RT}\frac{T}{T_s}\, dz^* = \frac{p}{RT_s}\, dz^* = \rho_0\, dz^* \,. \tag{4.71}$$

Since ρ_0 is proportional to p, it can be used to rewrite the continuity equation as

$$\frac{\partial \tilde{u}}{\partial x^*} + \frac{\partial \tilde{v}}{\partial y^*} + \frac{1}{\rho_0}\frac{\partial}{\partial z^*}\left(\rho_0 \tilde{w}\right) = 0 \,. \tag{4.72}$$

Since neither p nor ρ_0 depend on x^* or y^*, this becomes

$$\boxed{\nabla \cdot (p\, \tilde{\mathbf{u}}) = 0 \quad \text{or} \quad \nabla \cdot (\rho_0 \tilde{\mathbf{u}}) = 0}\,, \tag{4.73}$$

where the Nablaoperator is defined as

$$\nabla = \left(\frac{\partial}{\partial x^*}, \frac{\partial}{\partial y^*}, \frac{\partial}{\partial z^*}\right) \tag{4.74}$$

and where the components of the wind are combined into

$$\mathbf{u} = (\tilde{u}, \tilde{v}, \tilde{w}) \,. \tag{4.75}$$

Summarizing the above and dropping the tildes and asterisks, the primitive equations in ln-p coordinates read

$$\boxed{\begin{aligned}
&\frac{D\mathbf{v}}{Dt} + f\mathbf{k} \times \mathbf{v} = -\nabla_h \Phi + \mathbf{X}\,, && (a)\\
&\frac{\partial \Phi}{\partial z} = \frac{g}{T_s} T = \frac{g}{T_s}\, e^{-\frac{\kappa z}{H}}\, \theta\,, && (b)\\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho_0}\left(\rho_0 \frac{\partial w}{\partial z}\right) = 0\,, && (c)\\
&c_p \frac{D\theta}{Dt} = e^{\frac{\kappa z}{H}}\, J\,. && (d)
\end{aligned}} \tag{4.76}$$

with $D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$ and $\rho_0(z) = p(z)/(RT_s) = p_0\, \exp(-z/H)/(RT_s)$. With $\mathbf{X}$ and J given, the above are five equations for the five unknowns u, v, w, Φ and θ, all of them being functions of (x, y, z, t).

An alternative version for the heat equation is

$$\frac{D\theta}{Dt} = Q \tag{4.77}$$

with

$$Q = \frac{J}{c_p}\, e^{\frac{\kappa z}{H}} \; . \tag{4.78}$$

Using (4.60), we get

$$\frac{D\theta}{Dt} = \left(\frac{DT}{Dt} + \frac{\kappa}{H}\frac{Dz}{Dt}\right) e^{\frac{\kappa z}{H}} \tag{4.79}$$

and (4.76d) becomes

$$\frac{DT}{Dt} + \frac{\kappa}{H} w = \frac{J}{c_p} \tag{4.80}$$

or

$$\boxed{\frac{D_h T}{Dt} + \left(\frac{\partial T}{\partial z} + \frac{\kappa}{H} T\right) w = \frac{J}{c_p}} \tag{4.81}$$

with $D_h/Dt = \partial_t + \mathbf{v}\cdot\nabla_h$. Sometimes this is formulated as

$$\frac{D_h T}{Dt} + S\, w = \frac{J}{c_p} \; , \tag{4.82}$$

where the stability parameter S is defined as

$$S := \frac{\partial T}{\partial z} + \frac{\kappa}{H} T \; . \tag{4.83}$$

At this point we introduce the square of the *Brunt-Väisälä frequency* as measure for static stability. With geometric height z_g as vertical coordinate, we have

$$N_g^2 = \frac{g}{\theta}\frac{\partial\theta}{\partial z_g} \; . \tag{4.84}$$

In log-pressure coordinates the following modification turns out to be the natural measure of static stability[2]:

$$N^2 := N_g^2 \left(\frac{T}{T_s}\right)^2 \tag{4.85}$$

and we obtain

$$N^2 = \frac{g}{\theta}\frac{\partial\theta}{\partial z}\,\frac{\partial z}{\partial z_g}\frac{T^2}{T_s^2} = \frac{T}{T_s}\frac{g}{\theta}\frac{\partial\theta}{\partial z} = \frac{g}{T_s} e^{-\kappa z/H}\frac{\partial\theta}{\partial z} \; , \tag{4.86}$$

and we get the following relation between N^2, T, and θ in ln-p coordinates

$$\boxed{N^2 = \frac{g}{T_s}\left(\frac{\partial T}{\partial z} + \frac{\kappa}{H} T\right) = \frac{g}{T_s} S = \frac{g}{T_s}\frac{\partial\theta}{\partial z}\, e^{-\kappa z/H}} \; . \tag{4.87}$$

[2]Strictly speaking, the quantity N as defined in (4.85) is not the Brund-Väisälä frequency, but the difference between N and the true Brund-Väisälä frequency N_g is small since $T \approx T_s$; see Gill (1982), page 184.

Ertel PV

Without derivation we give here the form of Ertel Potential Vorticity consistent with the set of approximations inherent in the primitive equations. It reads

$$\boxed{P = \frac{1}{\rho_0}\, \boldsymbol{\zeta}_a \cdot \nabla\theta}\,, \tag{4.88}$$

with $\rho_0 = p/(RT_s)$, with absolute vorticity

$$\boldsymbol{\zeta}_a = \nabla \times \mathbf{v} + f\mathbf{k}\,, \tag{4.89}$$

with the horizontal wind $\mathbf{v} = u\mathbf{i} + v\mathbf{j}$, with the unit vector $\mathbf{k}$ in the vertical, and with $f = 2\Omega \sin\phi$ denoting the Coriolis parameter. Note that in the above expression horizontal derivatives are taken with z and, hence, p kept constant. The material rate of change if Ertel PV is given by

$$\boxed{\frac{DP}{Dt} = \frac{1}{\rho_0}\, \boldsymbol{\zeta}_a \cdot \nabla Q + \frac{1}{\rho_0}\, \nabla \times \mathbf{X} \cdot \nabla\theta}\,, \tag{4.90}$$

where $\mathbf{X}$ and Q are the nonconservative terms in the equations for horizontal momentum $\mathbf{v}$ and potential temperature θ. It follows that Ertel PV is materially conserved for adiabatic ($Q = 0$) frictionless ($\mathbf{X} = \mathbf{0}$) flow.

Vertical wind

We noted before that the the primitive equations do not contain an explicit equation for the vertical wind w, but nevertheless w is one of the variables and needs to be known in order to compute the material derivative $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$.

One way to determine w is via the continuity equation, which can be rewritten as

$$\partial_z(\rho_0 w) = -\nabla_h \cdot (\rho_0 \mathbf{v})\,. \tag{4.91}$$

Integrating in the vertical from z_0 to z and dividing by $\rho_0(z)$ gives

$$\begin{aligned} w(z) &= w(z_0)\, \frac{\rho_0(z_0)}{\rho_0(z)} - \frac{1}{\rho_0(z)} \int_{z_0}^{z} \nabla_h \cdot \left[\rho_0(z')\mathbf{v}(z')\right]\, dz' = \\ &= w(z_0)\, e^{(z-z_0)/H} - \frac{1}{\rho_0(z)} \int_{z_0}^{z} \nabla_h \cdot \left[\rho_0(z')\mathbf{v}(z')\right]\, dz' \end{aligned} \tag{4.92}$$

Keeping z fixed and leting $z_0 \to \infty$, the first term on the right hand side vanishes and we obtain

$$\boxed{w(z) = \frac{1}{\rho_0(z)} \int_{z}^{\infty} \nabla_h \cdot (\rho_0\, \mathbf{v})\, dz' = \int_{z_0}^{z} e^{(z-z')/H}\, \nabla_h \cdot \mathbf{v}\, dz'}\,. \tag{4.93}$$

The vertical wind in ln-p coordinates at level z is given through the horizontal mass divergence in the column above that level. This is perfectly consistent with the $w \propto$

Dp/Dt (see (4.67)) and the interpretation of pressure in a hydrostatic atmosphere being a measure of mass in the column overhead (see (4.9)).

Although (4.93) yields, in principle, a method to compute w, it is difficult to apply in practice. For synoptic and larger scales the wind is geostrophically balanced ($\mathbf{v} \approx \mathbf{v}_g$) to a good approximation. Since $\nabla_h \cdot \mathbf{v}_g = 0$, the actual wind $\mathbf{v}$ must be determined with very high accuracy because the net divergence is a small residuum of two terms which almost cancel each other. In fact, since $\mathbf{v} = \mathbf{v}_g + \mathbf{v}_a$ and $\nabla_h \cdot \mathbf{v}_g = 0$, it follows that $\nabla_h \cdot \mathbf{v} = \nabla_h \cdot \mathbf{v}_a$, i.e. it is the divergence of the ageostrophic wind which effectively appears in the integral on the right hand side of (4.93). As we shall see later, there are better ways to diagnose the vertical wind for balanced flows (see ω-equation).

References

Dutton, J. A., 1995: *Dynamics of Atmospheric Motion*. Dover Publications, Inc., 617 pp.

Gill, A. E., 1982: *Atmosphere–Ocean Dynamics*. Academic Press, New York, 662 pp.