Chapter 6

Three-Dimensional Quasi-Geostrophic Theory

The primitive equations are subject to scaling assuming small Rossby and Froude number. This leads to quasi-geostrophic theory. The key quantity is the quasi-geostrophic potential vorticity containing all the relevant information regarding the geostrophically balanced flow.

In this chapter we derive the equations for quasi-geostrophic flow on the β-plane. It is also possible to derive equations for quasi-geostrophic flow on a hemisphere, but this is more involved and not subject of this lecture. As it turns out, use the ln-p coordinates makes the equations particularly simple. So our starting point is the set of equations (4.76).

6.1 Quasi-geostrophic scaling

First we choose a barotropic static reference atmosphere characterized by $\Phi_0(z)$ and $T_0(z)$ such that hydrostatic balance is satisfied, i.e.

$$\frac{d\Phi_0}{dz} = \frac{g}{T_s} T_0 \ . \tag{6.1}$$

The corresponding reference potential temperature is

$$\theta_0(z) = T_0(z)\, e^{\kappa z/H} \ . \tag{6.2}$$

The fields are then written as the sum of this reference state plus a deviation, i.e.

$$\theta(x,y,z,t) = \theta_0(z) + \theta_e(x,y,z,t) \tag{6.3}$$

where the subscript e denotes the deviation from the reference state, and similarly for T and Φ. The actual variables which we shall deal with are T_e, θ_e and Φ_e, i.e. we deal with the deviations from the reference state, and this has to be kept in mind. In a certain sense (to be specified later) the deviation from the reference state needs to be small. It is, therefore, reasonable to choose a typical state as reference state which roughly represents the broad features of the atmosphere to be modeled. Referring to (4.87), where we introduced N^2 as a measure for static stability, the static stability of the reference state is characterized by

$$N_0^2(z) = \frac{g}{T_s}\left(\frac{\partial T_0}{\partial z} + \frac{\kappa}{H}T_0\right) = \frac{g}{T_s}S_0 = \frac{g}{T_s}\, e^{-\kappa z/H}\,\theta_{0z}\ , \tag{6.4}$$

where $S_0 := \partial T_0/\partial z + \kappa T_0/H$ and where we introduced the notation $\theta_{0z} := d\theta_0/dz$.

The scales we introduce are L for x and y, D for z, U for u and v, UD/L for w and L/U for t. As in the case of the shallow water equations (section 2.2), every variable is written as the product of its scale times a dimensionless variable, like for instance in the case of the zonal wind

$$u(\mathbf{x}, t) = U\, u^*(\mathbf{x}, t)\ , \tag{6.5}$$

where $|u^*| = \mathcal{O}(1)$. Using the Rossby number

$$Ro = \frac{U}{f_0 L}\ , \tag{6.6}$$

the zonal momentum equations becomes

$$Ro\left[\frac{\partial u^*}{\partial t^*} + (\mathbf{u}^*\cdot\nabla^*)\,u^*\right] - v^* - \frac{\beta L}{f_0}y^*v^* = -\frac{1}{f_0 LU}\frac{\partial \Phi}{\partial x^*}\ . \tag{6.7}$$

As before, the term $\beta L/f_0$ can be rewritten as

$$\frac{\beta L}{f_0} = \gamma Ro \tag{6.8}$$

with $\gamma = \beta L/(f_0 Ro)$, and assuming synoptic scale flow with $\Delta f = \beta L \ll f_0$ implies $\gamma \lesssim 1$. Since we want to obtain geostrophic balance to leading order in Ro, this determines the scale of geopotential as $f_0 LU$. We are left with the following dimensionless equations for horizontal momentum

$$\boxed{\begin{aligned} Ro\left[\frac{\partial u^*}{\partial t^*} + (\mathbf{u}^*\cdot\nabla^*)\,u^* - \gamma\, y^*v^*\right] - v^* &= -\frac{\partial \Phi^*}{\partial x^*}\ , && (a)\\ Ro\left[\frac{\partial v^*}{\partial t^*} + (\mathbf{u}^*\cdot\nabla^*)\,v^* + \gamma\, y^*u^*\right] + u^* &= -\frac{\partial \Phi^*}{\partial y^*}\ . && (b)\end{aligned}} \tag{6.9}$$

The continuity equation simply becomes

$$\boxed{\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} + \frac{1}{\rho_0}\frac{\partial}{\partial z^*}(\rho_0 w^*) = 0}\ , \tag{6.10}$$

where the scale W for the vertical wind was chosen to be

$$W = U\frac{D}{L}\ . \tag{6.11}$$

We shall see later that the vertical wind is in fact much smaller than suggested by W, which means that w^* is not $\mathcal{O}(1)$ but order Ro.

Next we try to find the appropriate scale for temperature (more precisely: temperature deviation from the reference state). We have already chosen the scale for geopotential. Using the hydrostatic equation

$$T_e = \mathcal{T} T^* = \frac{T_s}{g}\frac{\partial \Phi_e}{\partial z} = \frac{T_s}{g}\frac{f_0 LU}{D}\frac{\partial \Phi_e^*}{\partial z^*} , \tag{6.12}$$

where $\mathcal{T}$ denotes the scale for the temperature (deviation from the referenc state), this gives us

$$\mathcal{T} = T_s \frac{f_0 LU}{gD} , \tag{6.13}$$

such that the dimensionless version of the hydrostatic equation becomes

$$\boxed{\frac{\partial \Phi_e^*}{\partial z^*} = T_e^* = \left(\frac{p^*}{p_0^*}\right)^{\kappa} \theta_e^*} . \tag{6.14}$$

Care must be used when scaling the heat equation. For conservative flow, this is

$$\frac{\partial \theta}{\partial t} + \mathbf{v} \cdot \nabla_h \, \theta + w \frac{\partial \theta}{\partial z} = 0 . \tag{6.15}$$

Regarding the change of θ in the horizontal direction and with time, the relevant scale is $\mathcal{T}$ determined above, but we do not now at this point how to scale $\partial\theta/\partial z$. We write

$$\frac{\mathcal{T} U}{L}\left(\frac{\partial \theta^*}{\partial t^*} + \mathbf{v}^* \cdot {\nabla_h}^* \, \theta^*\right) + \frac{UD}{L} w^* \, \frac{\partial \theta}{\partial z} = 0 \tag{6.16}$$

or, after division by $(f_0 \mathcal{T})$,

$$Ro\left(\frac{\partial \theta^*}{\partial t^*} + \mathbf{v}^* \cdot {\nabla_h}^* \, \theta^*\right) + Ro \frac{D}{\mathcal{T}} \, w^* \, \frac{\partial \theta}{\partial z} = 0 . \tag{6.17}$$

We now make the additional assumption[1] that

$$\boxed{\frac{\partial \theta_e / \partial z}{d\theta_0 / dz} = \mathcal{O}(Ro) \ll 1} , \tag{6.18}$$

i.e. that the stable stratification is dominated by the reference state and that the deviation θ_e contributes to $\mathcal{O}(Ro)$ only. Observed distributions of $\theta(\mathbf{x})$ show that the assumption (6.18) is often not very well satisfied on the synoptic scale, and this limits the quantitative application of quasi-geostrophic theory to the Earth's atmosphere. However, we cannot proceed without making this assumption, and it is at least consistent with the other approximations. As it turns out, 3D quasi-geostrophic theory gives

[1]The goal of this additional assumption is to make the vertical wind w vanish to leading order in the Rossby number. In other words, at leading order we want to recover the geostrophic wind, which does not have a vertical component.

results which are qualitatively very useful, and this justifies our approach a posteriori. We thus can write

$$\frac{\partial \theta_e}{\partial z} \ll \frac{d\theta_0}{dz} \tag{6.19}$$

and, therefore,

$$\frac{\partial \theta}{\partial z} \approx \frac{d\theta_0}{dz} = \frac{1}{Ro}\frac{\partial \theta_e}{\partial z} = \frac{1}{Ro}\frac{\mathcal{T}}{D}\frac{\partial \theta^*}{\partial z^*}\,. \tag{6.20}$$

With this, the dimensionless heat equation reads

$$\boxed{Ro\left(\frac{\partial \theta^*}{\partial t^*} + \mathbf{v}^* \cdot \nabla_h{}^* \,\theta^*\right) + w^*\,\frac{\partial \theta^*}{\partial z^*} = 0}\,. \tag{6.21}$$

Note, incidentally, that according to (6.20) the limit of vanishing Rossby number ($Ro \to 0$) corresponds to a reference atmosphere with very high static stability ($N_0^2 \propto \theta_{0z} \to \infty$). This means that (6.18) effectively corresponds to the assumption of small Froude number (cf. chapter 2 for the corresponding development in the framework of the shallow water system). Remember that the Froude number for a continuously stratified fluid was introduced in chapter **??** as $Fr = U/(ND)$ (where D denotes the vertical scale of the motion), so given D and U large values of N indicats small values of Fr.

Formally, the dimensionless variables are expanded into an asymptotic series in Ro,

$$\begin{pmatrix} u^* \\ v^* \\ w^* \\ \Phi^* \\ \theta^* \end{pmatrix} = \begin{pmatrix} u^*_{(0)} \\ v^*_{(0)} \\ w^*_{(0)} \\ \Phi^*_{(0)} \\ \theta^*_{(0)} \end{pmatrix} + Ro \begin{pmatrix} u^*_{(1)} \\ v^*_{(1)} \\ w^*_{(1)} \\ \Phi^*_{(1)} \\ \theta^*_{(1)} \end{pmatrix} + Ro^2 \begin{pmatrix} u^*_{(2)} \\ v^*_{(2)} \\ w^*_{(2)} \\ \Phi^*_{(2)} \\ \theta^*_{(2)} \end{pmatrix} + \ldots\,. \tag{6.22}$$

To lowest order in Ro we get

$$-v^*_{(0)} = -\frac{\partial \Phi^*_{(0)}}{\partial x^*}\,, \tag{6.23}$$

$$u^*_{(0)} = -\frac{\partial \Phi^*_{(0)}}{\partial y^*}\,, \tag{6.24}$$

$$w^*_{(0)} = 0\,, \tag{6.25}$$

$$\frac{\partial u^*_{(0)}}{\partial x^*} + \frac{\partial v^*_{(0)}}{\partial y^*} = 0\,, \tag{6.26}$$

$$\frac{\partial \Phi^*_{(0)}}{\partial z^*} = T^*_{(0)} = \left(\frac{p^*}{p^*_0}\right)^{\kappa} \theta^*_{(0)}\,. \tag{6.27}$$

These equations are degenerate, since the fourth one is a simple consequence of the first two. Reintroducing scales, we call the first order contribution to the horizontal wind the geostrophic wind, i.e.

$$\mathbf{v}_g \equiv \begin{pmatrix} u_g \\ v_g \end{pmatrix} := \begin{pmatrix} Uu^*_{(0)} \\ Uv^*_{(0)} \end{pmatrix}\,. \tag{6.28}$$

In addition, we use the following notation[2]

$$\Phi_e = f_0 LU \, \Phi^*_{(0)} \tag{6.29}$$

and

$$T_e = \mathcal{T} \, T^*_{(0)} \; . \tag{6.30}$$

We get

$$\boxed{\begin{array}{ll} -f_0 v_g = -\dfrac{\partial \Phi}{\partial x} \, , & (a) \\ f_0 u_g = -\dfrac{\partial \Phi}{\partial y} \, , & (b) \\ \nabla_h \cdot \mathbf{v}_g = 0 \, , & (c) \\ \dfrac{\partial \Phi}{\partial z} = \dfrac{g}{T_s} T \, , & (d) \end{array}} \tag{6.31}$$

plus the fact that the vertical wind vanishes to leading order in Ro. Note that we returned to the reference state geopotential and temperature, which is possible since the horizontal gradient of Φ_0 vanishes. The first two equations describe geostrophic balance; this is by no means surprising, rather we constructed our scales such as to obtain this result. In vector notation we can write

$$\mathbf{v}_g = \frac{1}{f_0} \mathbf{k} \times \nabla_h \Phi \tag{6.32}$$

or

$$\mathbf{v}_g = \mathbf{k} \times \nabla_h \psi \quad \text{or} \quad \begin{pmatrix} u_g \\ v_g \end{pmatrix} = \begin{pmatrix} -\partial_y \psi \\ \partial_x \psi \end{pmatrix} \, , \tag{6.33}$$

with

$$\boxed{\psi := \frac{1}{f_0} \Phi_e \equiv \frac{1}{f_0} (\Phi - \Phi_0)} \tag{6.34}$$

denoting the *geostrophic stream function*. We have found the stream function for the geostrophic wind is essentially given by the geopotential.

The equations $\mathcal{O}(Ro)^1$ are

$$v^*_{(1)} - \frac{\partial \Phi^*_{(1)}}{\partial x^*} = D^*_{(0)} u^*_{(0)} - \gamma y^* v^*_{(0)} \, , \tag{6.35}$$

$$-u^*_{(1)} - \frac{\partial \Phi^*_{(1)}}{\partial y^*} = D^*_{(0)} v^*_{(0)} + \gamma y^* u^*_{(0)} \, , \tag{6.36}$$

$$\frac{\partial \Phi^*_{(1)}}{\partial z^*} = T^*_{(1)} \, , \tag{6.37}$$

$$\frac{\partial u^*_{(1)}}{\partial x^*} + \frac{\partial v^*_{(1)}}{\partial y^*} + \frac{1}{\rho_0} \frac{\partial}{\partial z^*} \left(\rho_0 \, w^*_{(1)} \right) = 0 \, , \tag{6.38}$$

$$-w^*_{(1)} \frac{\partial \theta^*_{(0)}}{\partial z^*} = D^*_{(0)} \theta^*_{(0)} \, , \tag{6.39}$$

[2] In contrast to the wind, we only need the lowest order contribution for geopotential and temperature and, therefore, do not introduce extra notation with subscript g.

where $D^*_{(0)} = \partial/\partial t + \mathbf{v}^*_{(0)} \cdot \nabla_h$. Reintroducing scales (detailes are given in the appendix to this chapter), this becomes

$$\begin{aligned} D_g u_g - f_0 v_a - \beta y v_g &= -\partial_x \Phi_a \, , & (6.40)\\ D_g v_g + f_0 u_a + \beta y u_g &= -\partial_y \Phi_a \, , & (6.41)\\ \frac{\partial \Phi_a}{\partial z} &= T_a \, , & (6.42)\\ \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 \, w_a) &= 0 \, , & (6.43)\\ D_g \theta_e + w_a \frac{d\theta_0}{dz} &= 0 \, . & (6.44) \end{aligned}$$

As in the case of the shallow water equations we have less equations than unknowns, but the evolution of the lowest order variables only depends on the horizontal divergence of the ageostrophic wind, which gives us the freedom to effectively set $\Phi_a = 0$. It follows that we do not need the hydrostatic equation at $\mathcal{O}(Ro)^1$. Assuming that nonconservative terms come into play at $\mathcal{O}(Ro)$, the equations finally read

$$\boxed{\begin{aligned} & D_g u_g - f_0 v_a - \beta y v_g = X \, , && (a)\\ & D_g v_g + f_0 u_a + \beta y u_g = Y \, , && (b)\\ & \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{1}{\rho_0}\frac{\partial}{\partial z}(\rho_0 \, w_a) = 0 \, , && (c)\\ & D_g \theta_e + w_a \frac{d\theta_0}{dz} = Q \, , && (d) \end{aligned}} \qquad (6.45)$$

where

$$D_g = \frac{\partial}{\partial t} + \mathbf{v}_g \cdot \nabla_h \qquad (6.46)$$

denotes the material derivative when following the geostrophic flow, $\mathbf{X} = (X, Y)$ is the external momentum forcing due to, e.g., friction and Q denotes the diabatic heating expressed for potential temperature in units $\mathrm{K\,s^{-1}}$.

The heat equation can be formulated in terms of T instead of θ, i.e. we could depart from (4.81) rather than (4.76d). Equation (6.45d) must then be replaced by

$$\boxed{D_g T_e + w_a S_0 = \tilde{Q}} \, , \qquad (6.47)$$

where $\tilde{Q} = Q \, \exp(-\kappa z/H)$ is the diabatic heating in terms of a material rate of change of T, and $S_0 = \theta_{0z} \exp(-\kappa z/H)$ (see (6.4)). Note that (6.47) is not strictly identical to (6.45d); yet, the two equations are equivalent in the sense that they have the same asymptotics as long as the actual static stability can be approximated by the static stability of the reference atmosphere.

6.2 Quasi-geostrophic potential vorticity

Forming $\partial_x(6.45b) - \partial_y(6.45a)$ gives

$$D_g(\partial_x v_g - \partial_y u_g) + (\partial_x \mathbf{v}_g \cdot \nabla)\, v_g - (\partial_y \mathbf{v}_g \cdot \nabla)\, u_g + f_0(\partial_x u_a + \partial_y v_a)$$

$$+\beta y(\partial_x u_g + \partial_y v_g) + \beta v_g = \partial_x Y - \partial_y X \ . \tag{6.48}$$

The second, third and fifth term on the left hand side vanish owing to (6.31c). Since $f = f_0 + \beta y$, the last term on the left hand side can be rewritten as $\beta v_g = D_g f$. The fourth term on the left is rewritten using (6.45c) such that we finally obtain

$$\boxed{D_g \zeta_g = \frac{f_0}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w_a) + \partial_x Y - \partial_y X} \ , \tag{6.49}$$

where the *geostrophic (absolute) vorticity* is defined as

$$\boxed{\zeta_g = f + \partial_x v_g - \partial_y u_g} \ , \tag{6.50}$$

corresponding to vertical component of absolute vorticity with the wind replaced by the geostrophic wind. Equation(6.49) is called *geostrophic vorticity equation*. It tells us that (absolute) geostrophic vorticity can be changed either through nonconservative processes or through divergence of the vertical wind. The latter corresponds to the process of vortex stretching, which we have encountered several times before.

Forming $\rho_0(6.45d)/(d\theta_0/dz)$, and noting that θ_0 depends neigther on t nor on x and y, gives us

$$D_g\left(\rho_0\frac{\theta}{\theta_{0z}}\right) + \rho_0 w_a = \rho_0\frac{Q}{\theta_{0z}}, \tag{6.51}$$

where we set $\theta_{0z} := d\theta_0/dz$ as an abbreviation. Forming the $f_0\rho_0^{-1}\partial\ldots/\partial z$, we get

$$D_g\frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{\theta}{\theta_{0z}}\right) + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}(\rho_0 w_a) = \frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{Q}{\theta_{0z}}\right) \ , \tag{6.52}$$

where we used that

$$(\partial_z \mathbf{v}_g \cdot \nabla)\left(\rho_0\frac{\theta}{\theta_{0z}}\right) = 0 \ , \tag{6.53}$$

which immediately follows from the $\mathcal{O}(Ro^0)$-equations. Equation (6.52) can be used to eliminate the ageostrophic vertical wind w_a from the geostrophic vorticity equation, yielding

$$D_g\left[\zeta_g + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{\theta_e}{\theta_{0z}}\right)\right] = \partial_x Y - \partial_y X + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{Q}{\theta_{0z}}\right) \tag{6.54}$$

Introducing *quasi-geostrophic potential vorticity* (often abbreviated as *quasi-geostrophic PV*) q_g as

$$\boxed{q_g := \zeta_g + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{\theta}{\theta_{0z}}\right) = f_0 + \beta y + \partial_x v_g - \partial_y u_g + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{\theta}{\theta_{0z}}\right)} \ , \tag{6.55}$$

and combining the nonconservative terms into

$$S_g := \partial_x Y - \partial_y X + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{Q}{\theta_{0z}}\right) \ , \tag{6.56}$$

the above result can be written as

$$\boxed{D_g q_g = S_g} \ , \tag{6.57}$$

which is the *quasi-geostrophic potential vorticity equation* An important special case is conservative flow (i.e. $\mathbf{X} = \mathbf{0}$ and $Q = 0$), in which case be get

$$\boxed{D_g q_g = 0 \quad \text{for conservative flow}} \, . \tag{6.58}$$

Quasi-geostrophic PV is materially conserved when following the geostrophic flow. To distinghish this result from the related result for Ertel-PV, which is materially conserved when following the full 3D wind, the quantity q_g is sometimes called *pseudo potential vorticity*.

Quasi-geostrophic PV only contains $\mathcal{O}(Ro^0)$ variables. Equation (6.57) thus gives us the evolution of the geostrophic variables in terms of geostrophic variables alone.

It is possible to reduce this equation to one equation for the *geostrophic stream function*, which was introduced earlier in equation (6.34). From (6.45c) we get

$$\boxed{T_e = \frac{T_s}{g}\frac{\partial \Phi_e}{\partial z} = f_0 \frac{T_s}{g}\frac{\partial \psi}{\partial z}} \quad \text{or} \quad \boxed{\theta_e = f_0 \frac{T_s}{g}\, e^{\kappa z/H}\, \frac{\partial \psi}{\partial z}} \, , \tag{6.59}$$

which relates measures of temperature to the vertical derivative of the stream function. Clearly this is a new element in our three-dimensional theory which transcends quasi-geostrophic shallow water theory, where neither temperature nor vertical derivatives can be properly defined. Earlier in (6.4) we found that

$$\theta_{0z} = N_0^2 \frac{T_s}{g}\, e^{\kappa z/H} \, . \tag{6.60}$$

We thus obtain

$$\frac{\theta_e}{\theta_{0z}} = \frac{f_0}{N^2}\, \frac{\partial \psi}{\partial z} \, , \tag{6.61}$$

allowing us to rewrite the quasi-geostrophic PV entirely in terms of ψ as

$$\boxed{q_g = f_0 + \beta y + {\nabla_h}^2 \psi + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{f_0^2}{N_0^2}\frac{\partial \psi}{\partial z}\right)} \, . \tag{6.62}$$

PV inversion

Assume that at some specific time the distribution of quasi-geostrophic PV $q_g(\mathbf{x})$ is given and known. Rearranging terms, (6.62) becomes

$$\boxed{\mathcal{L}\, \psi = q_g - f} \tag{6.63}$$

with the "generalized Laplacian"

$$\mathcal{L}\psi := \nabla_h^2 \psi + \frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0 \frac{f_0^2}{N_0^2}\frac{\partial \psi}{\partial z}\right) \, . \tag{6.64}$$

This can be viewed as an elliptic partial differential equation for the geostrophic stream function ψ. Assuming that appropriate boundary conditions are specified, this equation can be solved for ψ, from which we can obtain complete information about the balanced flow, i.e.

$$\begin{aligned} \mathbf{v}_g &= \mathbf{k} \times \nabla_h \psi \ , && (6.65) \\ T_e &= \frac{T_s}{g} f_0 \frac{\partial \psi}{\partial z} \ , && (6.66) \\ T &= T_0 + T_e \ , && (6.67) \\ \Phi &= \Phi_0 + \frac{g}{T_s} \int_0^z T_e(z')\, dz' \ . && (6.68) \end{aligned}$$

It is worth considering the *boundary conditions* for a moment. For a periodic channel we do not have boundaries in x, but there are "walls" at the northern and southern boundary and there is the surface bounding the atmosphere below. At the northern and southern boundary we specify $v = 0$ corresponding to $\partial_x \psi = 0$. At the lower boundary we specify the temperature, and for $z \to \infty$ we require that the stream function remains bounded. Taken together this is

$$\boxed{\begin{aligned} &\partial_x \psi = 0 \quad \text{at the northern and southern boundary} && (a) \\ &\partial_z \psi = \frac{g}{T_s f_0}(T - T_0) \equiv \frac{g}{T_s f_0} T_e \quad \text{at the lower boundary ,} && (b) \\ &|\psi| \ \text{ bounded } \ \text{ for } \ z \to \infty \ . && (c) \end{aligned}} \qquad (6.69)$$

We have homogeneous Neumann conditions north and south and (generally) inhomogeneous Neumann conditions at the lower boudary.

For later reference we note that the geostrophic relative vorticity is given in terms of the geostrophic stream function as

$$\zeta_{g,rel} = \zeta_g - f = \partial_x v_g - \partial_y u_g = \nabla_h^2 \psi \ , \qquad (6.70)$$

and furthermore

$$N_e^2 = N^2 - N_0^2 = f_0 \, e^{-\kappa z/H} \frac{\partial}{\partial z} \left(e^{\kappa z/H} \frac{\partial \psi}{\partial z} \right) \ . \qquad (6.71)$$

Bretherton's trick

We assume a flat surface with the atmosphere extending into the upper half space. When the actual temperature T deviates from the reference temperature T_0 at the lower boundary $z = 0$, we have a surface temperature anomaly $T_e \neq 0$ implying an inhomogeneous lower boundary condition for the inversion problem according to (6.69b). However, as was shown by Bretherton (1966), it is possible to replace the inhomogeneous lower boundary condition by a homogeneous one, when at the same time the interior PV is modified in an appropriate manner. This trick proves helpful later on to derive theorems, and furthermore it sheds an interesting light onto the role of surface temperature anomalies.

The problem to be solved is

$$\mathcal{L}\psi = q_g - f \quad \text{for} \quad z > 0 \tag{6.72}$$

with the *inhomogeneous* boundary condition

$$\frac{\partial \psi}{\partial z} = \frac{g}{f_0 T_s} T_e \quad \text{at} \quad z = 0 \, . \tag{6.73}$$

As we shall see shortly, this problem is equivalent to the following problem

$$\mathcal{L}\psi = \tilde{q}_g - f \quad \text{for} \quad z > 0 \tag{6.74}$$

with the *homogeneous* boundary condition

$$\frac{\partial \psi}{\partial z} = 0 \quad \text{at} \quad z = 0 \, , \tag{6.75}$$

where the *modified PV* is given by

$$\boxed{\tilde{q}_g := q_g + \frac{f_0^2}{N_0^2} \frac{g}{f_0 T_s} \, T_e \, \delta(z - 0_+) \equiv q_g + \frac{f_0}{S_0} \, T_e \, \delta(z - 0_+)} \, . \tag{6.76}$$

The modification has the structure of a delta function just above the lower boundary (this "just above" is denoted by 0_+). In order to see the equivalence, we integrate (6.74) in the vertical from 0 to a small positive value ϵ. In the limit of small ϵ and allowing a discontinuity of ψ_z, the only surviving terms are

$$\frac{f_0^2}{N_0^2} [\psi_z(\epsilon) - \psi_z(0)] = \int_0^\epsilon (\tilde{q}_g - f) \, dz \, . \tag{6.77}$$

The second term on the left hand side vanishes owing to (6.75), and we obtain

$$\frac{f_0^2}{N_0^2} \psi_z(0_+) = \int_0^\epsilon (q_g - f) \, dz + \frac{f_0^2}{N_0^2} \frac{g}{f_0 T_s} \, T_e \, . \tag{6.78}$$

Assuming that q_g is bounded and well behaved, the first term on the right hand side vanishes for $\epsilon \to 0$, and we are left with

$$\psi_z(0_+) = \frac{g}{f_0 T_s} \, T_e \, . \tag{6.79}$$

We have shown that the modification effectively produces the desired result, namely the "correct" inhomogeneous boundary condition (6.73) at $z = 0_+$. Furthermore, the modification of interior PV is restricted to the very shallow layer between 0 and 0_+, and the rest of the interior is unaffected. Taken together, this means that a delta-sheet of PV right above the surface with an appropriate amplitude is equivalent to a surface temperature anomaly T_e. Surface temperature anomalies and interior PV anomalies do not play fundamentally different roles in quasi-geostrophic theory. PV thinking applies to both, if one only remembers that a surface temperature anomaly corresponds to an infinitely thin PV anomaly right above the surface. For instance, if a poleward wind creates a local warm anomaly $T_e > 0$, while the PV in the interior is more or less homogeneous, we obtain a very shallow positive anomaly of $\tilde{q}_g$ in the northern hemisphere (or a negative anomaly of $\tilde{q}_g$ in the southern hemisphere). This PV anomaly is associated with cyclonic flow anomaly, which extends upward into the lower troposphere owing to the ellipticity of the inversion equation. This is why cyclogenesis often is associated with warm temperature advection at the surface, something which is well known from synoptic experience.

If we can assume $w_a = 0$ at the lower boundary, the equation for temperature or potential temperature at the lower boundary reduces to

$$D_g\theta = 0 \quad \text{or} \quad D_gT = 0 \quad \text{at} \quad z = 0 \; . \tag{6.80}$$

Combining this with $D_gq_g = 0$ in the interior of the domain, the evolution of modified PV is determined by

$$D_g\tilde{q}_g = 0 \; . \tag{6.81}$$

Ertel PV versus quasi-geostrophic PV

Ertel PV is defined as

$$P = \frac{1}{\rho_0}\,\boldsymbol{\zeta}_a \cdot \nabla\theta \; , \tag{6.82}$$

which is basically the *product* of absolute vorcitiy and static stability. Recall that in quasi-geostrophic theory the total potential temperature is split into two parts, $\theta(\mathbf{x},t) = \theta_0(z) + \theta_e(\mathbf{x},t)$, where θ_e denotes the deviation of total potential temperature from the reference atmosphere. Now, quasi-geostrophic PV is

$$q_g = \zeta_g + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{\theta_e}{\theta_{0z}}\right) = f + \zeta_{g,rel} + \frac{f_0}{\rho_0}\frac{\partial}{\partial z}\left(\rho_0\frac{\theta_e}{\theta_{0z}}\right) \; . \tag{6.83}$$

Quasi-geostrophic PV is basically the *sum* (rathern than the product) of absolute vorticity and static stability.

The question we want to address here is the following: does P asymptotically approach q_g if we assume quasigeostrophic scaling, i.e. do we get

$$P \sim q_g \quad \text{for} \quad Ro \to 0\;? \tag{6.84}$$

As we shall show now, the answer is: not quite.

Recall that for quasi-geostrophic scaling we have $|\zeta_{rel}| \ll |f_0|$ and $\partial_z\theta_e \ll \theta_{0z}$. We expand the dot product in the definition of P as

$$P = \frac{1}{\rho_0}(f\mathbf{k} + \nabla\times\mathbf{v})\cdot\nabla\theta = \frac{1}{\rho_0}(f\,\partial_z\theta + \zeta_{rel}\,\partial_z\theta - \partial_z v\,\partial_x\theta + \partial_z u\,\partial_y\theta) \; . \tag{6.85}$$

In this expression we write $\theta = \theta_0 + \theta_e$, and the wind is replaced by the geostrophic wind. This yields

$$P = \frac{1}{\rho_0}(f\partial_z\theta_0 + f\partial_z\theta_e + \zeta_{g,rel}\partial_z\theta_0 + \zeta_{g,rel}\partial_z\theta_e - \partial_z v_g\,\partial_x\theta_e + \partial_x u_g\,\partial_z\theta_e) \; . \tag{6.86}$$

The relative order of the terms in brackets on the right hand side regarding a series expansion in Ro is

$$(0,1,1,2,2,2) \; . \tag{6.87}$$

To lowest order we get

$$P(\mathbf{x},t) \sim P^{(0)}(z) = \frac{f_0\,\theta_{0z}}{\rho_0} \quad \text{for} \quad Ro \to 0\ . \tag{6.88}$$

Keeping the two highest orders we obtain

$$P \sim \frac{1}{\rho_0}(f\partial_z\theta_0 + f_0\partial_z\theta_e + \zeta_{g,rel}\partial_z\theta_0) = \frac{\theta_{0z}}{\rho_0}\left(f + \zeta_{g,rel} + f_0\frac{\partial_z\theta_e}{\theta_{0z}}\right)\ . \tag{6.89}$$

The term in brackets on the right hand side is similar, but not equivalent to the expression for q_g. From (6.55) we get

$$q_g = \left(f + \zeta_{g,rel} + f_0\frac{\partial_z\theta_e}{\theta_{0z}}\right) + \frac{f_0\theta_e}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{\theta_{0z}}\right)\ , \tag{6.90}$$

and, therefore,

$$P(\mathbf{x},t) \sim P^{(1)}(\mathbf{x},t) = \frac{\theta_{0z}}{\rho_0}\,q_g(\mathbf{x},t) - \frac{f_0\,\theta_e(\mathbf{x},t)}{\rho_0}\left[\frac{\theta_{0z}}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{\theta_{0z}}\right)\right] \quad \text{for} \quad Ro \to 0\ . \tag{6.91}$$

The second term on the right hand side vanishes only if $\partial_z(\rho_0/\theta_{0z}) = 0$. Generally, however, P does *not* asymptote $q_g\theta_{0z}/\rho_0$ for quasi-geostrophic scaling, and in this sense q_g *cannot* be viewed as an approximation to Ertel PV.

However, the situation is different regarding the material conservation of PV for conservative conditions. The material rate of change of Ertel PV is

$$\frac{DP}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_h\right)P + w\frac{\partial P}{\partial z}\ . \tag{6.92}$$

We set $\mathbf{v} \sim \mathbf{v}_g$ for quasi-geostrophic scaling and use (6.91) to subsitute for P in the first term on the right hand side. Since w vanishes to the lowest order in Ro, we set $w \sim w_a$ and use (6.88) in order to obtain the same order approximation for the second term on the right hand side. We get

$$\begin{aligned}
\frac{DP}{Dt} &= \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla_h\right)P + w\frac{\partial P}{\partial z} \sim D_gP^{(1)} + w_a\frac{\partial P^{(0)}}{\partial z} \sim \\
&\sim \frac{\theta_{0z}}{\rho_0}D_gq_g - \frac{f_0}{\rho_0}\frac{\theta_{0z}}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{\theta_{0z}}\right)D_g\theta_e + w_af_0\frac{\partial}{\partial z}\left(\frac{\theta_{0z}}{\rho_0}\right) = \\
&= \frac{\theta_{0z}}{\rho_0}D_gq_g + w_af_0\frac{\theta_{0z}^2}{\rho_0^2}\frac{\partial}{\partial z}\left(\frac{\rho_0}{\theta_{0z}}\right) + w_af_0\frac{\partial}{\partial z}\left(\frac{\theta_{0z}}{\rho_0}\right)\ ,
\end{aligned} \tag{6.93}$$

where we used the fact that the operator D_g does not involve vertical partial derivative, and where we used the conservative quasi-geostrophic heat equation to substitute $-w_a\theta_{0z}$ for $D_g\theta_e$. Noting the identity

$$\frac{\theta_{0z}^2}{\rho_0^2}\frac{\partial}{\partial z}\left(\frac{\rho_0}{\theta_{0z}}\right) = -\frac{\partial}{\partial z}\left(\frac{\theta_{0z}}{\rho_0}\right)\ , \tag{6.94}$$

the second and the third term on the right hand side cancel leaving us with

$$\frac{DP}{Dt} \sim \frac{\theta_{0z}}{\rho_0}D_gq_g \quad \text{for} \quad Ro \to 0 \quad \text{(for conservative flow)}\ . \tag{6.95}$$

For conservative flow, the statement of material conservation of P during advection with the three-dimensional wind $\mathbf{u}$ (i.e. $DP/Dt = 0$) asymptotically becomes the statement of material conservation of q_g during advection with the geostrophic wind $\mathbf{v}_g$ (i.e. $D_g q_g = 0$). This holds true despite the fact that P does *not* asymptote q_g for $Ro \to 0$.

With the help of (6.91) we find

$$\boxed{\left.\frac{\partial P}{\partial s}\right|_\theta \sim \frac{\theta_{0z}}{\rho_0}\left.\frac{\partial q_g}{\partial s}\right|_z \quad \text{for} \quad Ro \to 0 \quad (s = x, y, t)} \,, \tag{6.96}$$

showing us that horizontal gradients of Ertel PV in isentropic coordinates correspond to horizotal gradients of quasi-geostrophic PV in log-p coordinates. Results of quasi-geostrophic theory have a counterpart in more general balanced theory involving Ertel PV and its gradient in isentropic coordinates. It is no coincidence that the famous review article that initiated extensive work on the use of PV in atmospheric sciences is titled "On the use and significance of isentropic potential vorticity maps" (Hoskins et al. 1985).

6.3 Ageostrophic wind and Q-Vector

The quasi-geostrophic equations for conservative flow on the f-plane read

$$D_g u_g - f v_a = 0 \,, \tag{6.97}$$
$$D_g v_g + f u_a = 0 \,, \tag{6.98}$$
$$\partial_x u_a + \partial_y v_a + \frac{1}{\rho_0}\partial_z(\rho_0 w_a) = 0 \,, \tag{6.99}$$
$$D_g T_e + S_0 w_a = 0 \,, \tag{6.100}$$

where $T = T_0(z) + T_e$ and $S_0 = T_{0z} + \frac{\kappa}{H}T_0$ (see (6.4)). The geostrophic wind and the temperature deviation from the reference state are given by

$$f_0(u_g, v_g) = (-\partial_y\Phi, \partial_x\Phi) \,, \tag{6.101}$$
$$T_e = \frac{T_s}{g}\,\partial_z\Phi_e \,, \tag{6.102}$$

where $\Phi_e = \Phi - \Phi_0(z)$. Partial differentiation of (6.101) with respect to z and of (6.102) with respect to x and y yield the components of the thermal wind equation

$$f_0\,\partial_z u_g = -\partial^2_{yz}\Phi = -\frac{g}{T_s}\,\partial_y T \,, \tag{6.103}$$
$$f_0\,\partial_z v_g = \partial^2_{xz}\Phi = \frac{g}{T_s}\,\partial_x T \,, \tag{6.104}$$

relating the vertical shear of the geostrophic wind to the horizontal temperature gradient. In vector notation this is

$$\frac{g}{T_s}\nabla_h T = \begin{pmatrix} f_0\,\partial_z v_g \\ -f_0\,\partial_z u_g \end{pmatrix} = -f_0\mathbf{k}\times\mathbf{v}_g \,. \tag{6.105}$$

Equations (6.97) – (6.99) can be directly solved for the ageostrophic wind, yielding

$$\mathbf{v}_a = \frac{1}{f_0}\,\mathbf{k}\times D_g\mathbf{v}_g \ , \qquad (6.106)$$

$$w_a = -\frac{1}{S_0}\,D_gT_e \equiv -\frac{1}{S_0}\,D_gT \ . \qquad (6.107)$$

For instance, there is poleward ageostrophic flow in a jet-entrance region associated with the acceleration of the geostrophic wind following the geostrophic wind.[3] The above formulation involves explicit time derivatives and obscures the fact that the agostrophic wind is diagnostically related to the geostrohic wind and the temperature field. It is our goal to derive such a a purely diagnostic expression now.

Using (6.101) and (6.102) to subsitute for the geostrophic wind and temperature, equations (6.97), (6.98), and (6.100) can be rewritten as

$$D_g\,\partial_y\Phi = -f_0^2v_a \ , \qquad (6.109)$$

$$D_g\,\partial_x\Phi = -f_0^2u_a \ , \qquad (6.110)$$

$$D_g\,\partial_z\Phi = -\left(\frac{g}{T_s}S_0\right)w_a = -N_0^2w_a \ , \qquad (6.111)$$

where $N_0^2 = gT_s^{-1}S_0$ is a measure of the static stability of the reference atmosphere (see (6.4)). Forming $\partial_z(6.109) - \partial_y(6.111)$, we get

$$(\partial_z\mathbf{v}_g\cdot\nabla)\,\partial_y\Phi - (\partial_y\mathbf{v}_g\cdot\nabla)\,\partial_z\Phi = -f_0^2\,\partial_zv_a + N_0^2\,\partial_yw_a \ , \qquad (6.112)$$

or

$$N_0^2\,\partial_yw_a - f_0^2\,\partial_zv_a = -2\,\frac{g}{T_s}\,\partial_y\mathbf{v}_g\cdot\nabla T \ , \qquad (6.113)$$

where use was made of the thermal wind equation. Similarly, forming $\partial_z(6.110) - \partial_x(6.111)$ yields

$$(\partial_z\mathbf{v}_g\cdot\nabla)\,\partial_x\Phi - (\partial_x\mathbf{v}_g\cdot\nabla)\,\partial_z\Phi = -f_0^2\,\partial_zu_a + N_0^2\,\partial_xw_a \qquad (6.114)$$

or

$$N_0^2\,\partial_xw_a - f_0^2\,\partial_zu_a = -2\,\frac{g}{T_s}\,\partial_x\mathbf{v}_g\cdot\nabla T \ . \qquad (6.115)$$

Introducing the two-component *Q-vector*[4]

$$\boxed{\mathbf{Q} \equiv \begin{pmatrix} Q^{(1)} \\ Q^{(2)} \end{pmatrix} := -\frac{g}{T_a}\begin{pmatrix} \partial_x\mathbf{v}_g\cdot\nabla T \\ \partial_y\mathbf{v}_g\cdot\nabla T \end{pmatrix}} \ , \qquad (6.116)$$

[3] The equation for the ageostrophic horizontal wind is sometimes written as

$$\mathbf{v}_a = \frac{1}{f_0}\,\mathbf{k}\times\frac{\partial\mathbf{v}_g}{\partial t} + \frac{1}{f_0}\,\mathbf{k}\times(\mathbf{v}_g\cdot\nabla_h)\,\mathbf{v}_g \ , \qquad (6.108)$$

where the first term on the right hand side corresponds to the *isallobaric wind* and the second term contains nonlinear contributions. Close to the surface, the wind tends to be small such that the isallobaric wind sometimes is a good approximation to the ageostrophic wind, but in the interior of the atmosphere there may be substantial cancellation between the two terms such that the concept of the isallobaric wind is generally not very useful.

[4] In the next section we shall see that the $Q^{(1)}$ and $Q^{(2)}$ as defined in (6.116) are, indeed, the two components of a horizontal vector.

the above two equations can be written as

$$N_0^2\,\partial_y w_a - f_0^2\,\partial_z v_a \;=\; 2Q^{(2)}\;, \tag{6.117}$$

$$N_0^2\,\partial_x w_a - f_0^2\,\partial_z u_a \;=\; 2Q^{(1)}\;. \tag{6.118}$$

Forming $\partial_y(6.117) + \partial_x(6.118)$, we get

$$N_0^2\,\partial_{yy}^2 w_a - f_0^2\,\partial_{yz}^2 v_a + N_0^2\,\partial_{xx}^2 w_a - f_0^2\,\partial_{xz}^2 u_a = 2\,\nabla\cdot\mathbf{Q}\;. \tag{6.119}$$

Using (6.99), the second and fourth term on the left hand side can be combined into $f_0^2\partial_z[\rho_0^{-1}\partial_z(\rho_0 w_a)]$, yielding the final result

$$\boxed{N_0^2\,\nabla_h^2 w_a + f_0^2\,\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}\,(\rho_0 w_a)\right] = 2\,\nabla\cdot\mathbf{Q}} \tag{6.120}$$

or

$$\boxed{\mathcal{L}\,w_a = 2\,\nabla\cdot\mathbf{Q}}\;, \tag{6.121}$$

where the elliptic differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}\ldots = N_0^2\,\nabla_h^2\ldots + f_0^2\,\frac{\partial}{\partial z}\left[\frac{1}{\rho_0}\frac{\partial}{\partial z}\,(\rho_0\ldots)\right]\;. \tag{6.122}$$

This is the so-called *omega equation in Q-vector form.*[5] Several points are noteworthy:

- The omega equation *diagnostically* relates the ageostrophic vertical wind to the divergence of the Q-vector, and this require solving an elliptic partial differential equation. The forcing $\nabla\cdot\mathbf{Q}$ only involves the geostrophic wind and temperature and its derivatives, i.e. only quantities which can be diagnostically inferred from the fields Φ or q_g at any specific time. It can be computed from standard observations alone, which essentially provide $\Phi(\mathbf{x})$ at a certain time. The divergence of the Q-vector, $\nabla\cdot\mathbf{Q}$, is sometimes called *geostrophic forcing for the ageostrophic wind.*

- The operator $\mathcal{L}$ is similar to a three-dimensional Laplacian operator, but note that it differs from the operator relating q_g and ψ (see (6.63)). For a localized source $\nabla\cdot\mathbf{Q}$, the associated vertical wind w_a extends beyond the region of the source. Solving an elliptic partial differential equation involves an element of *non-locality.*

[5]The equation is called *omega-equation*, because historically it was first derived in pressure coordinates with the vertical wind denoted by ω,

$$\tilde{\mathcal{L}}\,\omega = A + B\;. \tag{6.123}$$

Here the operator $\tilde{\mathcal{L}}$ is the analogue to our operator $\mathcal{L}$, A represents differential vorticity advection, and B represents the advection of layer thickness. In this form the forcing is given as the sum of two terms A and B rather than the divergence of a vector. This has two major disadvantages. First, in contrast to $\nabla\cdot\mathbf{Q}$, the terms A and B are not invariant with respect to the choice of the coordinate system. Second (and that's worse) there is often substantial cancellation between A and B rendering the interpretation of the individual terms as mechanisms forcing the vertical ageostrophic wind problematic. The present form (6.120) involving the divergence of a vector on the right hand side is much preferred because it avoids such cancellations. See Hoskins et al. (1978) for a detailed discussion.

- For simple patterns of w_a we can assume that $\mathcal{L}\, w_a \propto -w_a$ gives an approximate indication for the *sign of the vertical wind.* It follows that convergence of the Q-vector forces ascent and divergence of the Q-vector forces descent.
- The omega equation is one of the more reliable methods to compute the vertical wind from observations.
- The Q-vector and, as a consequence, the forcing $\nabla \cdot \mathbf{Q}$ for the ageostrophic wind is nonlinear in geostrophic quantities. This has implications for the "attribution problem" , where one tries to isolate the impact of individual PV anomalies (see Wirth 2003). Symbolically, this fact can be written as

$$\mathcal{L} w_a = \mathcal{N}[\psi] \ , \tag{6.124}$$

where ψ is the geostrophic stream function and $\mathcal{N}[\ldots]$ denotes a nonlinear operator.

Interpretation of the Q-vector

We want to obtain a deeper understanding for the role of the ageostrophic wind. To this end we consider a hypothetical situation in which the ageostrohpic wind is switched off. In this hypothetical situation quation (6.100) becomes

$$D_g T = 0 \ . \tag{6.125}$$

Partial derivative with respect to x and y yields

$$\boxed{D_g \left(\frac{g}{T_s} \nabla T \right) = \mathbf{Q}} \ , \tag{6.126}$$

where ∇ denotes the horizontal gradient operator. This equation provides an interesting interpretation of the Q-vector. It indicates the rate of change of the temperature gradient when following the geostrophic wind in the hypothetical situation of no ageostrophic wind. Since horizontal temperature gradients are associated with fronts, the Q-vector quantifies the *geostrophic frontogenetic forcing.*

Similarly, for (hypothetically) $\mathbf{u}_a = \mathbf{0}$ equations (6.97) and (6.98) turn into

$$D_g u_g = 0 \ , \tag{6.127}$$

$$D_g v_g = 0 \ . \tag{6.128}$$

Partial derivative of the former of the two with respect to z yields

$$D_g(\partial_z u_g) + \partial_z \mathbf{v}_g \cdot \nabla u_g = 0 \tag{6.129}$$

or

$$\begin{aligned} D_g(f_0\, \partial_z u_g) &= -f_0\, \partial_z u_g\, \partial_x u_g - f_0\, \partial_z v_g\, \partial_y u_g = \\ = -\frac{g}{T_s}\, \partial_y T\, \partial_y v_g - \frac{g}{T_s}\, \partial_x T\, \partial_y u_g &= \frac{g}{T_s}\, \partial_y \mathbf{v}_g \cdot \nabla T = -Q^{(2)} \ , \end{aligned} \tag{6.130}$$

where use was made of the thermal wind equation and the nondivergence of the geostrohic wind. In a similar manner we obtain

$$D_g(f_0\,\partial_z v_g) = -Q^{(1)}\,. \tag{6.131}$$

Taken together, this is

$$D_g \begin{pmatrix} f_0\,\partial_z v_g \\ -f_0\,\partial_z u_g \end{pmatrix} = -\mathbf{Q}\,. \tag{6.132}$$

Now assume that the wind is balanced at initial time satisfying the thermal wind equation (6.105), repeated here for convenience

$$\frac{g}{T_s}\nabla T = \begin{pmatrix} f_0\,\partial_z v_g \\ -f_0\,\partial_z u_g \end{pmatrix}\,. \tag{6.133}$$

As we have shown, in the hypothetical case of no ageostrophic wind the rate of change of left hand side when following the geostrophic wind is given by $\mathbf{Q}$, while the rate of change of the right hand side when following the geostrophic wind is $-\mathbf{Q}$. Without ageostrophic wind the thermal wind would be destroyed during the ensueing evolution. If follows that the role of the ageostrophic wind is to keep the flow balanced throughout the evolution, which would be destroyed by the action of the geostrophic wind alone. Note the similarity to the Eliassen problem where the thermal wind balance of the primary circulation is disturbed by weak external forcing and where the ageostrophic secondary circulation is computed such as to guarantee thermal wind balance throughout the evolution.

Practical application of the Q-vector

Consider a map showing isolines of geopotential and temperature on a pressure surface in the middle troposphere. We want to compute the Q-Vector at point $\mathbf{x}$ on this map. It is useful to introduce a local cartesian coordinate system such that the x-axis is along the temperature isoline with the y-axis pointing towards lower temperatures, i.e. $\partial_x T = 0$ and $\nabla T = \partial_y T\,\mathbf{j}$ with $\partial_y T < 0$. The Q-vector reduces to

$$\mathbf{Q} = -\frac{g}{T_s}\begin{pmatrix} \partial_x v_g\,\partial_y T \\ \partial_y v_g\,\partial_y T \end{pmatrix} = \frac{g}{T_s}\,|\partial_y T| \begin{pmatrix} \partial_x v_g \\ -\partial_x u_g \end{pmatrix} = -\frac{g}{T_s}\,|\partial_y T|\,\mathbf{k}\times\partial_x\,\mathbf{v}_g\,. \tag{6.134}$$

We thus obtain the Q-vector through the following three steps:

1. determine $\partial_x\,\mathbf{v}_g$ as the vectorial change of $\mathbf{v}_g$ along the isotherm (with cold temperatures to the left),

2. rotate this vector by 90 ° in a clockwise sense (this corresponds to the operation $-\mathbf{k}\times\ldots$),

3. multiply the result by $|\partial_y T|$, i.e. by the strength of the temperature gradient.

The following figure illustrates the construction of the Q-vector in a jet entrance region on the Northern Hemisphere. In this case the Q-vector is directed from the cold add figure towards the warm wide of the jet. According to the omega equation, convergence of the Q-vector on the warm side is associated with rising motion and divergence on the cold side is associated with sinking motion. To ensure continuity of the ageostrophic wind, there must be poleward ageostrophic motion above and equatorward ageostrophic motion below, helping to accelerate and decelerate the wind in order to get it in line with the stronger temperature gradients in the jet streak region. At the same time the adiabatic warming and cooling reduces the temperature gradient such that the change of the geostrophic wind does not have to be as strong as without the ageostrophic wind. Like in the Eliassen problem the ageostrophic wind simultaneously affects both the wind and the temperature field in such a way that thermal wind balance is satisfied throughout the evolution despite the continuous "attempt" of the geostrophic wind to destroy thermal wind balance.

The ageostrophic vertical wind is upward on the right hand side of a jet entrance (and, similarly, on the left hand side of a jet exit), which facilitates surface cyclogenesis. This is an example for the fact that the balanced dynamics in the upper troposphere can have an indirect impact on the surface development through the ageostrophic circulation.

6.4 More general formulation

It is useful in 3D quasi-geostrophic theory to distinguish between the primary geostrophic flow with the associated temperature (represented by $\mathbf{v}_g$ and θ or T) and the secondary ageostrophic flow (represented by $\mathbf{u}_a$). We have shown in the previous sections that both can be diagnosed from the knowledge of q_g or Φ alone (assuming that suitable boundary conditions for PV inversion are provided). It is illuminating to consider the whole theory from a somewhat more mathematical point of view, as this will uncover a significant amount of structural elegance akin (but not identical) to that of Maxwell's equations of electrodynamics. We shall restrict our attention in this section to flow on the f-plane. The material in this section originates from the work of Nevir (1998).

The geostrophic equations (6.45a)–(6.45d) on the f-plane can be rewritten as

$$D_g u_g - f_0 v_a = X \,, \tag{6.135}$$

$$D_g v_g + f_0 u_a = Y \,, \tag{6.136}$$

$$D_g T + S_0 w_a = Q \,, \tag{6.137}$$

$$\nabla \cdot (\rho_0 \mathbf{u}_a) = 0 \,, \tag{6.138}$$

where X, Y, and Q denote the (otherwise unspecified) nonconservative terms. Using the partial derivatives of Φ to substitute for $\mathbf{v}_g$ and T shows that the first three equations are equivalent to

$$D_g \Phi_y + f_0^2 v_a = f_0 X \,, \tag{6.139}$$

$$D_g \Phi_x + f_0^2 u_a = f_0 Y \,, \tag{6.140}$$

$$D_g \Phi_z + N_0^2 w_a = \frac{g}{T_s} Q \,. \tag{6.141}$$

We define

$$\tilde{N}_0 := N_0 \,\frac{\rho_c}{\rho_0} \,, \tag{6.142}$$

where $\rho_c = \rho_0(0)$ is a constant density, and assume in the following that $N_0(z)$ is such that $\tilde{N}_0$ is constant, i.e.

$$\tilde{N}_0 = const \; . \tag{6.143}$$

Key to success turns out to be the use of a new vertical coordinate defined as

$$\tilde{z} = \frac{\tilde{N}_0}{f_0} \int^z \frac{1}{\rho_c} \rho_0(z')\, dz' \; . \tag{6.144}$$

It follows that

$$\frac{\rho_c}{\rho_0} \frac{\partial}{\partial z} = \frac{\tilde{N}_0}{f_0} \frac{\partial}{\partial \tilde{z}} \; , \tag{6.145}$$

the corresponding vertical velocity is

$$\tilde{w} = \frac{\tilde{N}_0}{f_0} \frac{\rho_0}{\rho_c} w \; , \tag{6.146}$$

and the quasi-geostrophic potential vorticity in terms of the stream function becomes

$$q_g = f_0 + \nabla_h^2 \psi + \frac{\partial^2 \psi}{\partial \tilde{z}^2} = f_0 + \tilde{\nabla}^2 \psi \; . \tag{6.147}$$

Equation (6.141) becomes

$$D_g \Phi_{\tilde{z}} + f_0^2 \tilde{w}_a = \frac{f_0}{N_0} \frac{g}{T_s} Q =: \tilde{Q} \; , \tag{6.148}$$

and the continuity equation becomes $\tilde{\nabla} \cdot \mathbf{u}_a = 0$. Dropping the tilde from now one, the new equations can be summarized as

$$\boxed{\begin{array}{ll} D_g \Phi_x + f_0^2 u_a = f_0 F^{(x)} \; , & (a) \\ D_g \Phi_y + f_0^2 v_a = f_0 F^{(y)} \; , & (b) \\ D_g \Phi_z + f_0^2 w_a = f_0 F^{(z)} \; , & (c) \\ \nabla \cdot \mathbf{u}_a = 0 & (d) \end{array}} \tag{6.149}$$

with the nonconservative terms written as

$$\mathbf{F} \equiv \begin{pmatrix} F^{(x)} \\ F^{(y)} \\ F^{(z)} \end{pmatrix} := \begin{pmatrix} Y \\ -X \\ \frac{1}{N_0} \frac{g}{T_s} \frac{Q}{\rho_0} \end{pmatrix} \; . \tag{6.150}$$

In vector notation, the above system becomes

$$D_g(\nabla \Phi) + f_0^2 \mathbf{u}_a = f_0 \mathbf{F} \; . \tag{6.151}$$

It immediately follows that the ageostrophic wind can be obtained from the konwledge of $\Phi(\mathbf{x}, t)$ as

$$\mathbf{u}_a = -D_g \left[\nabla (f_0^{-2} \Phi) \right] + f_0^{-1} \mathbf{F} \tag{6.152}$$

Although this is a perfectly valid way to compute $\mathbf{u}_a$, it is not a diagnostic relation; rather, it involves the partial time derivative $\partial_t \Phi$. What we are really interested in is a diagnostic relation between geostrophic quantities and the ageostrophic wind.

We define two vector fields. First, the vector field

$$\boxed{\mathbf{E}_g = \nabla \left(\frac{1}{f_0} \Phi \right)} , \tag{6.153}$$

which is the gradient of a scalar implying

$$\nabla \times \mathbf{E}_g = \mathbf{0} . \tag{6.154}$$

From this vector field the primary geostrophic flow can be obtained through

$$\begin{pmatrix} u_g \\ v_g \\ T \end{pmatrix} = \begin{pmatrix} -E_g^{(y)} \\ E_g^{(x)} \\ f_0 T_s g^{-1} E_g^{(z)} \end{pmatrix} . \tag{6.155}$$

In fact the divergence of $\mathbf{E}_g$ is related to q_g through

$$\nabla \cdot \mathbf{E}_g = \nabla^2 \left(\frac{1}{f_0} \Phi \right) = q_g - f_0 . \tag{6.156}$$

Second we define the vector

$$\boxed{\mathbf{B}_a := \mathbf{u}_a} \tag{6.157}$$

as simply the ageostrophic wind.

With this new notation, the geostrophic equations become

$$\boxed{\begin{aligned} &\partial_t \mathbf{E}_g + \frac{1}{f_0} J(\Phi, \mathbf{E}_g) + f_0 \mathbf{B}_a = f_0 \mathbf{F} , && (a) \\ &\nabla \cdot \mathbf{B}_a = 0 . && (b) \end{aligned}} \tag{6.158}$$

It is assumed that the vector $\mathbf{F}$ is given or can be computed from $\mathbf{E}_g$ and $\mathbf{B}_a$.

The dynamical evolution is obtained by forming the divergence of (6.158a) and using (6.158b) yielding

$$\partial_t (\nabla \cdot \mathbf{E}_g) + \frac{1}{f_0} J(\Phi, \nabla \cdot \mathbf{E}_g) = \nabla \cdot (f_0 \mathbf{F}) . \tag{6.159}$$

With (6.156), this becomes

$$\partial_t q_g + \frac{1}{f_0} J(\Phi, q_g) = \nabla \cdot (f_0 \mathbf{F}) , \tag{6.160}$$

which is nothing but the familiar qg PV equation

$$D_g q_g = \nabla \cdot (f_0 \mathbf{F}) . \tag{6.161}$$

On the other hand, computing the curl of (6.158a) and using (6.154) yields

$$\nabla \times \mathbf{B}_a = \nabla \times \left(-\frac{1}{f_0^2} J(\Phi, \mathbf{E}_g)\right) + \nabla \times \mathbf{F} \, . \tag{6.162}$$

Defining a new vector

$$\mathbf{N}_g := \nabla \times \left(-\frac{1}{f_0^2} J(\Phi, \mathbf{E}_g)\right) \, , \tag{6.163}$$

this becomes

$$\nabla \times \mathbf{B}_a = \mathbf{N}_g + \nabla \times \mathbf{F} \, . \tag{6.164}$$

The definition of $\mathbf{N}_g$ as the curl of a vector implies

$$\nabla \cdot \mathbf{N}_g = 0 \, . \tag{6.165}$$

A few lines of algebra show that $\mathbf{N}_g$ can be written in various forms, e.g.

$$\mathbf{N}_g = -\frac{1}{f_0} \nabla \times \begin{pmatrix} J(\Phi, \Phi_x) \\ J(\Phi, \Phi_y) \\ J(\Phi, \Phi_z) \end{pmatrix} = \frac{2}{f_0} \begin{pmatrix} J(\Phi_z, \Phi_y) \\ J(\Phi_x, \Phi_z) \\ J(\Phi_y, \Phi_x) \end{pmatrix} = -\frac{2}{f_0} \begin{pmatrix} \frac{\partial(u_g, v_g)}{\partial(y,z)} \\ \frac{\partial(u_g, v_g)}{\partial(z,x)} \\ \frac{\partial(u_g, v_g)}{\partial(x,y)} \end{pmatrix} \, . \tag{6.166}$$

In contrast to $\mathbf{E}_g$, the vector $\mathbf{N}_g$ depends nonlinearly on Φ, which is why it is abbreviated with a capital N.

Summarizing the above results, we have described the geostrophic primary flow through an irrotational vector field $\mathbf{E}_g$ the divergence of which is given by PV:

$$\boxed{\begin{aligned} &\nabla \times \mathbf{E}_g = \mathbf{0} \, , && (a) \\ &\nabla \cdot \mathbf{E}_g = q_g - f_0 \equiv \frac{1}{f_0} \nabla^2 \Phi \, . && (b) \end{aligned}} \tag{6.167}$$

The ageostrophic flow component $\mathbf{u}_a = \mathbf{B}_a$ is a solenoidal (i.e. zero divergence) vector field, the curl of which is the vector $\mathbf{N}_g$ plus a contribution from the nonconservative terms:

$$\boxed{\begin{aligned} &\nabla \cdot \mathbf{B}_a = 0 \, , && (a) \\ &\nabla \times \mathbf{B}_a = \mathbf{N}_g + \nabla \times \mathbf{F} \, . && (b) \end{aligned}} \tag{6.168}$$

From the point of view of PV thinking, the following set of equation appears convenient. First, the time evolution of q_g is obtained through

$$\boxed{D_g q_g = S_g} \, , \tag{6.169}$$

where $S_g = \nabla \cdot (f_0 \mathbf{F})$ contains the nonconservative terms. Given q_g at any instance, the geopotential Φ is obtained through solving the following Poisson equation, in which the quasi-geostrophic PV anomaly appears a forcing:

$$\boxed{\nabla^2 \Phi = f_0 (q_g - f_0) \, .} \tag{6.170}$$

Forming[6] $\nabla \times$ (6.168*b*) and accounting for (6.168a) yields

$$\boxed{\nabla^2 \mathbf{u}_a = -\nabla \times \mathbf{N}_g - \nabla \times (\nabla \times \mathbf{F})} . \qquad (6.171)$$

Thus, knowledge of Φ and, therfore, $\mathbf{N}_g$ plus the nonconservative terms $\mathbf{F}$ at one particular instance of time allows one to compute the right hand side of a Poisson equation for $\mathbf{u}_a$. Under conservative conditions both the geostrophic part of the flow (given by Φ) and the ageostrophic flow $\mathbf{u}_a$ are diagnostically related to q_g. Nonconservative terms give rise to an additional ageostrophic component.

The omega-equation can now be identified as a special case of the above equations. Projecting (6.171) onto the unit vector $\mathbf{k}$ in the vertical yields (in case of conservative flow)

$$\nabla^2 w_a = -\nabla \times \mathbf{N}_g \cdot \mathbf{k} \equiv \partial_x(-N_g^{(y)}) + \partial_y(N_g^{(x)}) . \qquad (6.172)$$

The right hand side is invariant regarding rotation in the horizontal plane, since $\mathbf{N}_g$ is a vector in three-dimensional space. Correspondingly, the right hand side can be written as the horizontal divergence of a horizontal vector,

$$\nabla^2 w_a = \nabla_h \cdot \tilde{\mathbf{Q}} , \qquad (6.173)$$

with

$$\tilde{\mathbf{Q}} \equiv \begin{pmatrix} \tilde{Q}^{(x)} \\ \tilde{Q}^{(y)} \end{pmatrix} := \begin{pmatrix} -N_g^{(y)} \\ N_g^{(x)} \end{pmatrix} = \mathbf{k} \times \mathbf{N}_g^{hor} , \qquad (6.174)$$

where $\mathbf{N}_g^{hor}$ is the horizontal projection of $\mathbf{N}_g$ and $\mathbf{k} \times \ldots$ corresponds to a rotation by 90 degrees. Indeed, multiplying the conventional omega equation (6.120) by ρ_c/ρ_0 and using our modified vertical coordinate, its left hand side becomes simply the Laplacian of w_a (i.e. $\mathcal{L} \to \nabla^2$), and a few further lines of algebra show that $\nabla_h \cdot \tilde{\mathbf{Q}} \to 2\nabla_h \cdot \mathbf{Q}$ with $\mathbf{Q}$ given by the earlier definition (6.116). This proves that (6.173) is identical to the more conventional form of the omega equation (6.120); furthermore it suggests that $Q^{(1)}$ and $Q^{(2)}$ are, indeed, the two components of a horizontal vector.

Apendix

This appendix shows how physical scales are introduced into the dimensionless equations. In the case of the dimensionless hydrostatic equation, $\partial\Phi^*_{(1)}/\partial z^* = T_{(1)}$, multiplication by $\mathcal{T}$ yields

$$T_s \frac{f_0 L U}{g D} \frac{\partial \Phi^*_{(1)}}{\partial z^*} = \mathcal{T} T_{(1)} \qquad (6.175)$$

from which one obtains $T_s g^{-1} \partial\Phi_a/\partial z = T_a$, where the subscript a denotes the dimensional quantity to first order in the Rossby number. Similarly, the dimensionless heat equation is multiplied by $u\mathcal{T}/L$ to give

$$D_g \theta + w^*_{(1)} \frac{\partial \theta^*_{(0)}}{\partial z^*} \frac{U}{L} \mathcal{T} = 0 . \qquad (6.176)$$

[6]Note that $\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$ for any vector $\mathbf{a}$.

Recalling that $\partial\theta^*_{(0)}/\partial z^* = \theta_{0z} Ro D/\mathcal{T}$ and $W = UD/L$, this becomes

$$D_g\theta + (w^*_{(1)} Ro W)\,\theta_{0z} = 0 \tag{6.177}$$

or

$$D_g\theta + w_a\theta_{0z} = 0\ , \tag{6.178}$$

with $w_a = Ro\,W w^*_{(1)}$ denoting the ageostrophic vertical wind.

References

Hoskins, B. J., I. Draghici, and H. C. Davies, 1978: A new look at the ω-equation. *Quart. J. Roy. Met. Soc.*, **104**, 31–38.

Hoskins, B. J., M. E. McIntyre, and A. W. Robertson, 1985: On the use and significance of isentropic potential vorticity maps. *Quart. J. Roy. Met. Soc.*, **111**, 877–946.

Névir, P., 1998: *Die Nambu-Felddarstellungen der Hydro-Thermodynamik und ihre Bedeutung für die dynamische Meteorologie.* Habilitation, Freie Universität Berlin.

Wirth, V., 2003: Potentielle Vorticity in der dynamischen Meteorologie. Skript zur Vorlesung.